# MATHEMATICAL HANDBOOK

CONTAINING

THE CHIEF FORMULAS OF ALGEBRA, TRIGONOMETRY, CIRCULAR AND HYPERBOLIC FUNCTIONS, DIFFERENTIAL AND INTEGRAL CALCULUS, AND ANALYTICAL GEOMETRY

TOGETHER WITH

*MATHEMATICAL TABLES*

SELECTED AND ARRANGED
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# PREFACE.

THE uses which this book may serve hardly need to be pointed out. Some years ago the writer composed the part relating to Trigonometry and used it as a syllabus for instruction in his college classes. It served its purpose and soon went out of print. But a stray copy of it found its way to the table of a well-known civil engineer, to whom it proved constantly useful, and by whom it was often referred to as "his memory." This engineer has suggested a revision and republication of the original book with important enlargements. Accordingly there have been added Sections on Algebra, the Differential and Integral Calculus, and Analytic Geometry. The subject of Hyperbolic Functions, which now receives much more attention than formerly, has been more fully treated. Tables have been added, which include not only those universally used, but also some — like those of the Hyperbolic Functions, of the Natural Logarithms of Numbers, and that of the Velocity of Falling Bodies ($v = 2\sqrt{gh}$) — that have been hitherto not readily accessible.

Of course no efforts have been spared to secure correctness in the printing of the formulas and the tables; but persons experienced in such work need not be reminded of the improbability that the first edition of a book of this kind should be absolutely free from error. The writer and the publishers can only add, that notice of any errors that may be detected will be thankfully received, and the necessary corrections will be promptly made and published. Also, suggestions of desirable additions to the book and of other improvements are invited with a view to their use in possible future editions.

E. P. S.

JUNE, 1907.

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# SECTION I.

## ALGEBRA.

### The General Laws of Common Algebra.

1. *The Law of Association.*

$$a+b+c=a+(b+c),$$
$$a+b-c=a+(b-c),$$
$$a-b+c=a-(b-c),$$
$$a-b-c=a-(b+c),$$

$$abc=a\,(bc)=(ab)\,c,$$

$$a\times b\div c=a\times(b\div c),$$
$$a\div b\div c=a\div(b\times c),$$
$$a\div b\times c=a\div(b\div c),$$

wherein the concurrence of *like* signs gives the *direct* sign $+$ or $\times$; and the concurrence of *unlike* signs gives the *indirect* sign $-$ or $\div$. Thus,

$$+(+c)=+c, \qquad \times(\times c)=\times c,$$
$$-(-c)=+c, \qquad \div(\div c)=\times c,$$
$$+(-c)=-c, \qquad \times(\div c)=\div c,$$
$$-(+c)=-c, \qquad \div(\times c)=\div c.$$

2. *The Law of Commutation.*

$$a+b=\quad b+a,$$
$$a-b=-b+a,$$

$$ab=ba,$$

$$a\times b\times c=a\times c\times b,$$
$$a\times b\div c=a\div c\times b,$$
$$a\div b\div c=a\div c\div b,$$
$$a\div b\times c=a\times c\div b.$$

3. *The Law of Distribution.*

For multiplication,

$$a(b+c) = ab + ac,$$

$$(\pm a \pm b) \times (\pm c \pm d) = +(\pm a) \times (\pm c) + (\pm a) \times (\pm d) + (\pm b) \times (\pm c) + (\pm b) \times (\pm d) = \pm ac \pm ad \pm bc \pm bd,$$

wherein the signs of each partial product are determined by the following rule:

If a partial product has factors with like signs, it must have the sign $+$; if factors with unlike signs, it must have the sign $-$. Thus,

$$+(+a) \times (+c) = +ac, \qquad +(+a) \times (-c) = -ac,$$
$$+(-a) \times (-c) = +ac, \qquad +(-a) \times (+c) = -ac.$$

For division,

$$(\pm a \pm b) \div (\pm c) = +(\pm a) \div (\pm c) + (\pm b) \div (\pm c),$$

with the following rule for signs:

If the dividend and divisor of a partial quotient have like signs, the partial quotient must have the sign $+$; if they have unlike signs, it must have the sign $-$. Thus,

$$+(+a) \div (+c) = +(a \div c), \qquad +(+a) \div (-c) = -(a \div c),$$
$$+(-a) \div (-c) = +(a \div c), \qquad +(-a) \div (+c) = -(a \div c).$$

Otherwise expressed, this law is

$$\frac{\pm a \pm b}{\pm c} = +\frac{\pm a}{\pm c} + \frac{\pm b}{\pm c},$$

with the same rule for signs; that is,

$$+\left(\frac{+a}{+c}\right) = +\frac{a}{c}, \qquad +\left(\frac{+a}{-c}\right) = -\frac{a}{c},$$

$$+\left(\frac{-a}{-c}\right) = +\frac{a}{c}, \qquad +\left(\frac{-a}{+c}\right) = -\frac{a}{c}.$$

The divisor cannot be distributed.

**Definitions and Laws of the Symbols**

$0, 1,$ *and* $\infty$.

4. $0 = +a - a = -a + a,$ $\qquad 1 = \times a \div a = \div a \times a,$

$\pm b + 0 = \pm b - 0,$ $\qquad \genfrac{}{}{0pt}{}{\times}{\div} a \times 1 = \genfrac{}{}{0pt}{}{\times}{\div} a \div 1,$

$+0 = -0.$ $\qquad \times 1 = \div 1.$

$0 \times (\pm b) = (\pm b) \times 0 = 0,$ $\qquad \infty \times (\pm b) = (\pm b) \times \infty = \pm \infty,$

$0 \div (\pm b) = 0,$ $\qquad \infty \div (\pm b) = \pm \infty,$

$$\left.\begin{array}{l} +b \div 0 = +\infty, \\ -b \div 0 = -\infty. \end{array}\right\} \qquad (\pm b) \div (\pm \infty) = 0.$$

5. Using $A$ and $B$ to represent any two algebraic expressions of quantity,

If $\quad A \times B = 0,\quad$ either $\quad A = 0$ or $B = 0,$
or both $\quad A = 0$ and $B = 0.$

If $\quad A \div B = 0$ and $B$ is not 0, then $A = 0.$

If $\quad A \div B = 0$ and $A$ is not 0, then $B = \infty.$

The forms $0 \times \infty$, $0 \div 0$ or $\frac{0}{0}$, $\infty \div \infty$ or $\frac{\infty}{\infty}$, and $\infty - \infty$ require special investigations to determine their values in the particular cases in which they arise. See pages 109, 110.

**Fractions and Ratios.**

6. Equivalent forms of notation,

$$a \div b = \frac{a}{b} = a : b = a/b.$$

7. Addition of fractions,

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$$

8. Subtraction of fractions,

$$\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}.$$

9. Multiplication of fractions,

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

10. Division of fractions,

$$\left.\begin{aligned}\frac{a}{b}\div\frac{c}{d}&=\frac{a}{b}\times\frac{d}{c}\\&=\frac{a\div c}{b\div d}\end{aligned}\right\}=\frac{ad}{bc}.$$

## Proportions.

11. If $a : b = c : d$, then $ad = bc$.

12. If $ad = bc$, then

$$\begin{aligned}a : b &= c : d,\\ b : a &= d : c,\\ a : c &= b : d,\\ c : a &= d : b;\end{aligned}$$

also,

$$\begin{aligned}&a \pm c : b \pm d = a : b = c : d,\\ &a + c : a - c = b + d : b - d,\\ &a + b : a - b = c + d : c - d,\\ &a \pm b : a = c \pm d : c,\\ &a \pm b : b = c \pm d : d;\end{aligned}$$

also,

$$\begin{aligned}ma : mb &= nc : nd,\\ \frac{a}{m} : \frac{b}{m} &= \frac{c}{n} : \frac{d}{n},\\ ma : nb &= mc : nd,\\ \frac{a}{m} : \frac{b}{n} &= \frac{c}{m} : \frac{d}{n}.\end{aligned}$$

13. If $a : b = c : d$, and $p : q = r : s$,
then $ap : bq = cr : ds$.

14. If $a : b = c : d$,
then

$$\begin{aligned}a^2 : b^2 &= c^2 : d^2,\\ a^n : b^n &= c^n : d^n,\\ \sqrt[n]{a} : \sqrt[n]{b} &= \sqrt[n]{c} : \sqrt[n]{d}.\end{aligned}$$

15. If $a : b = c : x$, then $x = \frac{bc}{a}$.

16. If $a : b = x : d$, then $x = \frac{ad}{b}$.

17. If
$$\frac{a}{A}=\frac{b}{B}=\frac{c}{C}=\dots$$
then
$$\frac{a}{A}=\frac{a+b+c+\dots}{A+B+C+\dots},$$
$$=\frac{pa+qb+rc+\dots}{pA+qB+rC+\dots}.$$

18. If $a:b=b:c$, then $b=\sqrt{ac}$, one geometric mean between $a$ and $c$.

19. If $a:b=b:c=c:d$, then $b=\sqrt[3]{a^2d}$ and $c=\sqrt[3]{ad^2}$, two geometric means between $a$ and $d$.

20. The *reciprocal* of $a$ is $\frac{1}{a}=a^{-1}$,

of $\frac{a}{b}$ is $\frac{b}{a}=\left(\frac{a}{b}\right)^{-1}$,

of $\frac{1}{a}$ is $a=\left(\frac{1}{a}\right)^{-1}$.

21. If $a:b=\frac{1}{p}:\frac{1}{q}$, then $p$ and $q$ are *inversely* or *reciprocally proportional* to $a$ and $b$; and the proportion may be written
$$a:b=q:p,$$
or
$$a:b=p^{-1}:q^{-1}.$$

22. If $x$ varies as $y$ *directly*, then
$$x_1:x_2=y_1:y_2$$
wherein $x_1, y_1$ and $x_2, y_2$ denote simultaneous or corresponding values of the variables $x$ and $y$.

23. If $x$ varies as $y$ *inversely*, then
$$x_1:x_2=\frac{1}{y_1}:\frac{1}{y_2}$$
or
$$x_1:x_2=y_2:y_1.$$

For example, the force of gravitation, $g$, varies inversely as the square of the distance, $d^2$, that is
$$g_1:g_2=\frac{1}{d_1^2}:\frac{1}{d_2^2}=d_2^2:d_1^2.$$

## Powers.

24. $(+a)^n = +a^n$.

25. $(-a)^{2n} = +a^{2n}$.

26. $(-a)^{2n+1} = -a^{2n+1}$.

27. $a^m \times a^n = a^{m+n}$.

28. $a^m \div a^n = a^{m-n}$.

29. $a^m \div a^m = a^0 = 1$.

30. $a^0 \div a^n = a^{-n} = \dfrac{1}{a^n}$.

31. $a^m \times a^{-n} = a^m \div a^n$.

32. $a^m \div a^{-n} = a^m \times a^n$.

33. $a^0 \div a^{-n} = a^n = \dfrac{1}{a^{-n}}$.

34. $(a^m)^n = a^{mn} = (a^n)^m$.

35. $(ab)^m = a^m b^m$.

36. $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$.

37. $\left(\dfrac{a}{b}\right)^{-n} = \dfrac{a^{-n}}{b^{-n}} = \dfrac{b^n}{a^n} = \left(\dfrac{b}{a}\right)^n$.

38. $0^n = 0$.

39. $0^{-n} = \infty$.

40. $\infty^n = \infty$.

41. $\infty^{-n} = 0$.

42. $a^{+0} = 1$.

43. $a^{-0} = 1$.

44. If $a > 1$, then $a^\infty = \infty$, and $a^{-\infty} = 0$.

45. If $a < 1$, then $a^\infty = 0$, and $a^{-\infty} = \infty$.

46. $\log 0 = -\infty$.

47. $\log 1 = 0$.

48. $\log \infty = \infty$.

The forms $0^0$, $1^\infty$, $\infty^0$ require special investigation. See page 109.

## Products and Factors.

49. $a^2 - b^2 = (a-b)\,(a+b)$.

50. $a^3 - b^3 = (a-b)\,(a^2 + ab + b^2)$.

51. $a^3 + b^3 = (a+b)\,(a^2 - ab + b^2)$.

52. $a^n - b^n =$
$(a-b)\,(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + b^{n-1})$, always.

53. $a^n - b^n = (a+b)(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \ldots - b^{n-1})$, if $n$ be even.

54. $a^n + b^n = (a+b)(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \ldots + b^{n-1})$, if $n$ be odd.

55. $(x+a)(x+b) = x^2 + (a+b)\,x + ab$.

56. $(x + a)(x + b)(x + c) = x^3 + (a + b + c)\,x^2$
$\qquad + (ab + bc + ca)\,x + abc.$

57. $(x + a)(x + b)(x + c)(x + d) = x^4 + (a + b + c + d)x^3$
$\qquad + (ab + ac + ad + bc + bd + cd)\,x^2$
$\qquad + (abc + abd + acd + bcd)x$
$\qquad + abcd.$

58. $(a + b)^2 = a^2 + 2ab + b^2 = a^2 + b^2 + 2ab.$

59. $(a - b)^2 = a^2 - 2ab + b^2 = a^2 + b^2 - 2ab.$

60. $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + b^3 + 3ab(a + b).$

61. $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3 = a^3 - b^3 - 3ab(a - b).$

For the general formula giving any power of a binomial, see 78 to 82.

62. *To square a polynomial.* Square each term and add to this square twice the product of that term by every term that follows it. Thus,

$$(a + b + c + d + e)^2 =$$
$$a^2 + 2a(b + c + d + e) + b^2 + 2b(c + d + e) + c^2 + 2c(d + e) + d^2 + 2de + e^2,$$
$$(a + b - c)^2 = a^2 + 2a(b - c) + b^2 - 2bc + c^2,$$
$$(a - b - c)^2 = a^2 - 2a(b + c) + b^2 + 2bc + c^2.$$

63. $a^4 + a^2b^2 + b^4 = (a^2 + ab + b^2)(a^2 - ab + b^2).$

64. $a^4 + b^4 = (a^2 + ab\sqrt{2} + b^2)(a^2 - ab\sqrt{2} + b^2).$

65. $\left(a + \dfrac{1}{a}\right)^2 = a^2 + \dfrac{1}{a^2} + 2.$

66. $\left(a + \dfrac{1}{a}\right)^3 = a^3 + \dfrac{1}{a^3} + 3\left(a + \dfrac{1}{a}\right).$

67. $(a + b + c)^3 =$
$a^3 + b^3 + c^3 + 3(b^2c + bc^2 + c^2a + ca^2 + a^2b + ab^2) + 6abc.$

68. $a^2 + b^2 - c^2 + 2ab = (a + b)^2 - c^2,$
$\qquad = (a + b + c)(a + b - c).$

69. $a^2 - b^2 - c^2 + 2bc = a^2 - (b - c)^2,$
$\qquad = (a + b - c)(a - b + c).$

70. $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - bc - ca - ab).$

71. $bc^2 + b^2c + ca^2 + c^2a + ab^2 + a^2b + a^3 + b^3 + c^3$
$= (a + b + c)(a^2 + b^2 + c^2).$

72. $bc^2 + b^2c + ca^2 + c^2a + ab^2 + a^2b + 3abc$
$= (a + b + c)(bc + ca + ab).$

73. $bc^2 + b^2c + ca^2 + c^2a + ab^2 + a^2b + 2abc$
$= (b + c)(c + a)(a + b).$

74. $bc^2 + b^2c + ca^2 + c^2a + ab^2 + a^2b - 2abc - a^3 - b^3 - c^3$
$= (b + c - a)(c + a - b)(a + b - c).$

75. $bc^2 - b^2c + ca^2 - c^2a + ab^2 - a^2b = (b - c)(c - a)(a - b).$

76. $2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4$
$= (a + b + c)(b + c - a)(c + a - b)(a + b - c).$

77. $a^3 + 2a^2b + 2ab^2 + b^3 = (a + b)(a^2 + ab + b^2).$

## The Binomial Theorem.

78. $(a + b)^n =$

$$a^n + \frac{n}{1}a^{n-1}b + \frac{n(n-1)}{1 \times 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 \times 2 \times 3}a^{n-3}b^3 + \ldots$$

wherein $n$ may be positive or negative, integral or fractional. When $n$ is a positive integer, the right hand member has $n + 1$ terms; when $n$ is negative or fractional, the number of terms is infinite.

79. The general expression for the $(r+1)^{\text{th}}$ term is

$$\frac{n(n-1)(n-2)(n-3)\ldots(n-r+1)}{1 \times 2 \times 3 \times \ldots r}a^{n-r}b^r,$$

or $$\frac{n(n-1)(n-2)\ldots\ldots 3 \times 2 \times 1}{1 \times 2 \times 3 \times \ldots r \times (n-r)(n-r-1) \times \ldots 2 \times 1}a^{n-r}b^r,$$

or, using the factorial notation,

$$\frac{n!}{r!\,(n-r)!}a^{n-r}b^r;$$

and the formula may be written

$$(a + b)^n = \sum_{r=0}^{r=n} \frac{n!}{r!\,(n-r)!}a^{n-r}b^r. \quad \text{[N. B. } 0! = 1.$$

**80.** The coefficients of the several terms in the expansion of the $n^{th}$ power of a binomial are conveniently designated by $C_0$, $C_1$, $C_2$, etc. These are functions of $n$ as follows:

$$C_0 = n^0 = 1,$$

$$C_1 = \frac{n}{1},$$

$$C_2 = \frac{n(n-1)}{1 \times 2},$$

$$C_3 = \frac{n(n-1)(n-2)}{1 \times 2 \times 3},$$

$$C_4 = \frac{n(n-1)(n-2)(n-3)}{1 \times 2 \times 3 \times 4}.$$

and in general

$$C_r = \frac{n(n-1)(n-2)\ldots(n-r+1)}{1 \times 2 \times 3 \times \ldots r}$$

$$= \frac{n!}{r!\,(n-r)!}.$$

Then

**81.** $(a+b)^n = C_0a^n + C_1a^{n-1}b + C_2a^{n-2}b^2 + C_3a^{n-3}b^3 + \ldots$

Also,

**82.** $(a-b)^n = C_0a^n - C_1a^{n-1}b + C_2a^{n-2}b^2 - C_3a^{n-3}b^3 + \ldots$

The numerical values of $C_1$, $C_2$, $C_3$, ... for each power of the binomial from the first ($n=1$) to the twentieth ($n=20$) power may be found in a table on page 244.

The numerical values of factorials from $n=0$ to $n=20$ may be found in a table on the same page.

## Inequalities.

**83.** The value of the fraction

$$\frac{a_1 + a_2 + a_3 + \ldots + a_n}{b_1 + b_2 + b_3 + \ldots + b_n}$$

is less than the greatest and greater than the least of the fractions $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots \frac{a_n}{b_n}$ provided the denominators of the latter are all positive.

**84.** The arithmetical mean of two numbers is greater than the geometrical, and the geometrical is greater than the harmonical. That is,

$$\frac{a+b}{2} > \sqrt{ab} > \frac{2ab}{a+b}.$$

Also,

**85.** $$\frac{a_1 + a_2 + \ldots + a_n}{n} > \sqrt[n]{a_1 a_2 \ldots a_n}.$$

The arithmetical mean of the powers is greater than the power of the arithmetical mean, that is,

**86.** $$\frac{a^m + b^m}{2} > \left(\frac{a+b}{2}\right)^m,$$

and, in general,

**87.** $$\frac{a_1{}^m + a_2{}^m + \ldots a_n{}^m}{n} > \left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right)^m,$$

excepting when $m$ is a positive proper fraction.

**88.** If $a$, $b$, $c$, be positive quantities,

$$a^a b^b > \left(\frac{a+b}{2}\right)^{a+b},$$

$$a^a b^b c^c > \left(\frac{a+b+c}{3}\right)^{a+b+c}.$$

**89.** If $m > n > a$,

$$\left(\frac{m+a}{m-a}\right)^m < \left(\frac{n+a}{n-a}\right)^n.$$

## Roots.

**90.** $a^{\frac{1}{m}} = \sqrt[m]{a}$.

**91.** $\sqrt[m]{a} \times \sqrt[n]{a} = a^{\frac{1}{m}} \times a^{\frac{1}{n}} = a^{\frac{m+n}{mn}} = \sqrt[mn]{a^{m+n}}$.

**92.** $\sqrt[n]{a} \div \sqrt[m]{a} = a^{\frac{1}{n}} \div a^{\frac{1}{m}} = a^{\frac{m-n}{mn}} = \sqrt[mn]{a^{m-n}}$.

**93.** $\left(a^{\frac{1}{m}}\right)^n = a^{\frac{n}{m}} = \left(\sqrt[m]{a}\right)^n = \sqrt[m]{a^n}$.

**94.** $\left(a^{\frac{1}{m}}\right)^m = \left(\sqrt[m]{a}\right)^m \sqrt[m]{a^m} = a$.

95. $\left(a^{\frac{1}{m}}\right)^{\frac{1}{n}} = a^{\frac{1}{mn}} = \sqrt[n]{\sqrt[m]{a}} = \sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}.$

96. $\sqrt[n]{a^{mn}} = a^{\frac{mn}{n}} = a^m.$

97. $\sqrt[n]{a^m} = \sqrt[kn]{a^{km}} = \sqrt[\frac{n}{k}]{a^{\frac{m}{k}}} = a^{\frac{km}{kn}} = a^{\frac{m}{n}}.$

98. $\sqrt[m]{ab} = \sqrt[m]{a} \times \sqrt[m]{b}.$

99. $(ab)^{\frac{1}{m}} = a^{\frac{1}{m}} b^{\frac{1}{m}}.$

100. $\sqrt[m]{\left(\frac{a}{b}\right)} = \frac{\sqrt[m]{a}}{\sqrt[m]{b}}.$

101. $\left(\frac{a}{b}\right)^{\frac{1}{m}} = \frac{a^{\frac{1}{m}}}{b^{\frac{1}{m}}}.$

102. $\sqrt[m]{\frac{1}{a}} = \frac{1}{\sqrt[m]{a}} = a^{-\frac{1}{m}}.$

Let $A$ represent a positive number, and $a$ the arithmetical value of its indicated root. Then,

103. $$\begin{cases} \sqrt[2n]{+A} = \pm a, & \sqrt[2n+1]{+A} = +a, \\ \sqrt[2n]{-A} = \pm ia, & \sqrt[2n+1]{-A} = -a, \\ \text{wherein } i = \sqrt{-1}. \end{cases}$$

## Surds.

104. If a number partly rational and partly surd is equal to another number also partly rational and partly surd, the two rational parts are equal and the two surd parts are equal. Thus if

$$a + \sqrt[n]{b} = x + \sqrt[n]{y},$$

wherein $a$ and $x$ are rational, and $\sqrt[n]{b}$ and $\sqrt[n]{y}$ are surds, then

$$a = x, \text{ and } b = y.$$

105. If $$\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y},$$

then $$\sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}.$$

106. $\sqrt{a} + \sqrt{b} = \sqrt{a + b + 2\sqrt{ab}}.$

107. $\sqrt{a} - \sqrt{b} = \sqrt{a + b - 2\sqrt{ab}}.$

108. $(a \pm \sqrt{b})^2 = a^2 + b \pm 2a\sqrt{b}.$

109. $(a + \sqrt{b})(a - \sqrt{b}) = a^2 - b.$

### The imaginary unit, $i$, and its powers.

110. By definition,

$$i = +\sqrt{-1},\quad i^2 = -1,\quad i^3 = -\sqrt{-1},\quad i^4 = +1.$$

111. Then,

$$i^5 = i^9 = i^{4n+1} = i;$$
$$i^6 = i^{10} = i^{4n+2} = i^2 = -1;$$

$$i^7 = i^{11} = i^{4n+3} = i^3 = -i;$$
$$i^8 = i^{12} = i^{4n} = i^4 = +1.$$

112. Also,

$$\frac{1}{i} = i^{-1} = i^3 = -i;$$

$$\frac{1}{i^3} = i^{-3} = i;$$

$$i^0 = +1.$$

### Complex Numbers.

113. A complex number is a collection of units partly real and partly imaginary. In its simplest form it is written $a \pm bi$ or $a \pm ib$, wherein $a$ denotes the number of real units and $b$ the number of imaginary units in the collection. Both $a$ and $b$ are real coefficients, the first of 1 the second of $i$.

114. If two complex numbers are equal their real parts are equal and their imaginary parts are equal.

Thus if $A + iB = a + ib$, then $A = a$ and $B = b$.

115. The two complex numbers $a + ib$ and $a - ib$ are *conjugates* the one of the other, and

$$(a + ib)(a - ib) = a^2 + b^2.$$

Product and quotient of two complex numbers.

116. $$(a + ib)(c + id) = ac - bd + i(bc + ad).$$

117. $$\frac{a + ib}{c + id} = \frac{ac + bd}{c^2 + d^2} + i\,\frac{(bc - ad)}{c^2 + d^2}.$$

**118.** Every complex number can be brought into the form

$$a \pm ib = r(\cos\theta \pm i\sin\theta)$$

wherein $\quad r = \sqrt{a^2 + b^2}$ = the *modulus*,

$$\theta = \tan^{-1}\frac{b}{a} = \cos^{-1}\frac{a}{r} = \sin^{-1}\frac{b}{r} = \text{the } argument.$$

**119.** The product of two complex numbers is found by multiplying their moduli and adding their arguments. Thus,

$$r_1(\cos\theta_1 \pm i\sin\theta_1) \times r_2(\cos\theta_2 \pm i\sin\theta_2) =$$
$$r_1r_2[\cos(\theta_1 + \theta_2) \pm i\sin(\theta_1 + \theta_2)]$$

**120.** The quotient is found by dividing one modulus by the other and subtracting the argument of the divisor from that of the dividend. Thus,

$$\frac{r_1(\cos\theta_1 \pm i\sin\theta_1)}{r_2(\cos\theta_2 \pm i\sin\theta_2)} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) \pm i\sin(\theta_1 - \theta_2)].$$

**121.** Powers of a complex number.

$$[r(\cos\theta + i\sin\theta)]^m = r^m(\cos m\theta + i\sin m\theta).$$

**122.** Roots of a complex number.

$$\left[r(\cos\theta + i\sin\theta)\right]^{\frac{m}{n}} = r^{\frac{m}{n}}\left[\cos\frac{m\theta + 2k\pi}{n} + i\sin\frac{m\theta + 2k\pi}{n}\right].$$

Relations of conjugates.

**123.** $(a + ib)(a - ib) = r^2.$

**124.** $\begin{cases} a + ib = r(\cos\theta + i\sin\theta) = re^{i\theta}. \\ a - ib = r(\cos\theta - i\sin\theta) = re^{-i\theta}. \end{cases}$

**125.** $(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = 1.$

Roots of 1 and of $-1$.

**126.** $$\sqrt[n]{1} = \cos\frac{2k\pi}{n} \pm i\sin\frac{2k\pi}{n}.$$

**127.** $$\sqrt[n]{-1} = \cos\frac{(2k + 1)\pi}{n} \pm i\sin\frac{(2k + 1)\pi}{n}.$$

## Logarithms.

The relation between a number, $x$, and its logarithm, $u$, is expressed by the equations

128. $$x = a^u, \qquad u = \log_a x$$

wherein $a$ is the base of the system of logarithms intended.

The relations between logarithms of the same number in systems having different bases are thus shown.

If $x = a^u$, then $u = \log_a x$;

and if $x = b^v$, then $v = \log_b x$.

Whence
$$1 = a^u \div b^v,$$
$$0 = u - v \log_a b,$$
$$0 = u \log_b a - v,$$

129. $$\log_a x = \log_b x \times \log_a b,$$

130. $$\log_b x = \log_a x \times \log_b a,$$

131. $$\log_a b \times \log_b a = 1.$$

The two systems of logarithms most in use are the Natural System, founded upon the Exponential Base, $e$, and the Common System, founded upon the base 10. Logarithms of the former system are often called hyperbolic logarithms or Naperian logarithms, and those of the latter Briggs' logarithms or denary logarithms

Writing 10 for $a$ and $e$ for $b$, the foregoing equations become

132. $$\log_{10} x = \log_e x \times \log_{10} e,$$

133. $$\log_e x = \log_{10} x \times \log_e 10,$$

134. $$\log_{10} e \times \log_e 10 = 1.$$

The Exponential Base, $e$, is the limit of $\left(1 + \frac{1}{m}\right)^m$ as $m$ approaches $\infty$.

135. $$e = 1 + \frac{1}{1} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$$
$$= 2.718\ 281\ 828\ 459 \ldots$$

$M = \log_{10} e$ is known as the *modulus* of the Common System of logarithms. Any logarithm in the Natural System multiplied by the modulus gives the corresponding logarithm in the Common System; and, conversely, any logarithm in the Common System divided by the modulus gives the corresponding logarithm in the Natural System. Thus,

**136.** $\log_{10} x = M \times \log_e x, \quad \log_e x = M^{-1} \times \log_{10} x.$

**137.** $M = \log_{10} e = 0.434\ 294\ 481\ 903 \ldots$

**138.** $M^{-1} = \log_e 10 = 2.302\ 585\ 092\ 994 \ldots$

**139.**

| | Multiples of $M$. | | Multiples of $M^{-1}$. |
|---|---|---|---|
| 1 | 0.434 294 481 903 | 1 | 2.302 585 092 994 |
| 2 | 0.868 588 963 807 | 2 | 4.605 170 185 988 |
| 3 | 1.302 883 445 710 | 3 | 6.907 755 278 982 |
| 4 | 1.737 177 927 613 | 4 | 9.210 340 371 976 |
| 5 | 2.171 472 409 516 | 5 | 11.512 925 464 970 |
| 6 | 2.605 766 891 420 | 6 | 13.815 510 557 964 |
| 7 | 3.040 061 373 323 | 7 | 16.118 095 650 958 |
| 8 | 3.474 355 855 226 | 8 | 18.420 680 743 952 |
| 9 | 3.908 650 337 129 | 9 | 20.723 265 836 946 |

A positive number has an unlimited number of logarithms; but only one of these is real, namely the one obtained by giving to the arbitrary integer $k$ the value 0 in the following general equations:

**140.** $\log_e (+x) = \log_e x \pm 2k\pi i.$

**141.** $\log_e (+1) = 0 \pm 2k\pi i.$

Negative numbers have no *real* logarithms,

**142.** $\log_e (-x) = \log_e x \pm (2k+1)\pi i.$

**143.** $\log_e (-1) = 0 \pm (2k+1)\pi i.$

Complex numbers have complex logarithms,

**144.** $\log_e (a + ib) = \frac{1}{2}\log_e (a^2 + b^2) + i\left(\tan^{-1}\frac{b}{a} \pm k\pi\right).$

Imaginary numbers have imaginary logarithms,

145. $\log_e i = \frac{1}{2}\pi i$, 146. $i^i = e^{-\frac{1}{2}\pi} = 0.20788 \ldots$

Rules for the practical use of logarithms are based on the following principles:

147. $\log (xy) = \log x + \log y.$

148. $\log\left(\frac{x}{y}\right) = \log x - \log y.$

149. $\log (x^n) = n \log x.$

150. $\log \sqrt[n]{x} = \frac{1}{n} \log x.$

151. $\log \text{base} = 1, \quad \log 1 = 0, \quad \log 0 = -\infty.$

152.

$$\left\{\begin{array}{ll} \text{If} & 1 < x < +\infty, \\ \text{then,} & 0 < \log x < +\infty. \\ \text{If} & 0 < x < +1, \\ \text{then,} & -\infty < \log x < 0. \end{array}\right\} \ldots$$

That is, if $x$ is positive and greater than 1, its logarithm is positive; if positive and less than 1, its logarithm is negative.

## Permutations and Combinations.

153. The number of *permutations* (sometimes called arrangements) of $n$ things taken all at a time is

$$n(n-1)(n-2) \ldots 2 \times 1, \text{ or } n!$$

154. The number of permutations of $n$ things taken $r$ at a time may be denoted by the symbol $P(n, r)$.

$$P(n, r) = n(n-1)(n-2) \ldots \text{to } r \text{ factors},$$
$$= \frac{n!}{(n-r)!}.$$

155. The number of *combinations* of $n$ things taken $r$ at a time may be denoted by the symbol $C(n, r)$.

$$C(n, r) = \frac{n(n-1)(n-2) \ldots (n-r+1)}{1 \times 2 \times 3 \ldots r} = \frac{P(n, r)}{r!}$$
$$= \frac{n!}{r!(n-r)!} = C(n, n-r).$$

Comparing 155 with 79 it may be seen that

156. $C(n, r)$ = the binomial coefficient of the $(r+1)^{th}$ term of the development of $(a+b)^n$.

Numerical values of $C(n\ r)$ up to $n = 20$ are found in the table on page 244.

## Determinants.

If there be $n^2$ quantities whose symbols are arrayed in the form of a square of $n$ rows and $n$ columns, this array is the symbol of a determinant. The $n^2$ quantities forming the array are the elements of the determinant. The determinant itself is the algebraic sum of all the products that can be formed of $n$ elements taken one from each column and each row in all possible ways, one half of these products being written with the positive sign, the other half with the negative.

157. An array of four elements, the symbol of a determinant of the second order, gives $2!(=2)$ terms, thus:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$$

158. An array of nine elements, the symbol of a determinant of the third order, gives $3!\ (=6)$ terms, thus:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1.$$

$$\begin{array}{ccc} \left|\begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{matrix}\right| \\ a_1 \quad b_1 \quad c_1 \\ a_2 \quad b_2 \quad c_2 \end{array}$$

NOTE. — If the determinant array of the third order be written with the first two rows repeated as shown in the margin, then the positive terms of its development can be found by reading the three diagonal rows from the left downwards, and the negative terms by reading the three diagonal rows from the left upwards.

**159.** An array of sixteen elements, the symbol of a determinant of the fourth order, gives $4!(=24)$ terms, thus:.

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \begin{aligned} &= a_1b_2c_3d_4 - a_1b_2c_4d_3 + a_3b_1c_2d_4 - a_3b_1c_4d_2 \\ &+ a_1b_3c_4d_2 - a_1b_3c_2d_4 + a_3b_2c_4d_1 - a_3b_2c_1d_4 \\ &+ a_1b_4c_2d_3 - a_1b_4c_3d_2 + a_3b_4c_1d_2 - a_3b_4c_2d_1 \\ &- a_2b_1c_3d_4 + a_2b_1c_4d_3 - a_4b_1c_2d_3 + a_4b_1c_3d_2 \\ &- a_2b_3c_4d_1 + a_2b_3c_1d_4 - a_4b_2c_3d_1 + a_4b_2c_1d_3 \\ &- a_2b_4c_1d_3 + a_2b_4c_3d_1 - a_4b_3c_1d_2 + a_4b_3c_2d_1 \end{aligned}$$

An array of twenty-five elements, the symbol of a determinant of the fifth order, gives $5!(=120)$ terms. In general, a determinant of the $n^{\text{th}}$ order consists of $n!$ terms.

**160.** If the row and the column in which a given element stands be stricken out, the determinant formed of the remaining elements is the *minor determinant* relative to the given element.

If each element in one column of the major determinant be multiplied by its relative minor determinant and the positive sign be given to each element taken from an odd numbered row and the negative sign to each element taken from an even numbered row, the algebraic sum of the results is equal to the major determinant. Thus,

**161.**

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1).$$

**162.**

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_4 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}.$$

Thus can a determinant of any order, the $n^{\text{th}}$, be made to depend on $n$ determinants of the $(n-1)^{\text{th}}$ order, and

each of these again on $n-1$ determinants of the $(n-2)^{\text{th}}$ order, and so on, the ultimate result being that the original determinant depends on a series of determinants of the third or of the second order, which last are easily computed directly. This method of reduction makes easy the computation of the value of any determinant with numerical elements.

**163.** In any determinant the columns can be made rows and the rows columns without changing its value. Thus,

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}; \qquad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

**164.** If, in any determinant, two columns or two rows change places with each other, the new determinant so formed is equal to the first one with the opposite sign. Thus,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = \text{etc.}$$

**165.** If the elements in two columns or in two rows are equal or proportional each to each, the value of the determinant is 0.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0; \begin{vmatrix} a_1 & a_1 & b_1 \\ a_2 & a_2 & b_2 \\ a_3 & a_3 & b_3 \end{vmatrix} = 0; \begin{vmatrix} a_1 & na_1 & b_1 \\ a_2 & na_2 & b_2 \\ a_3 & na_3 & b_3 \end{vmatrix} = n\begin{vmatrix} a_1 & a_1 & b_1 \\ a_2 & a_2 & b_2 \\ a_3 & a_3 & b_3 \end{vmatrix} = n0 = 0.$$

**166.** A determinant is multiplied or divided by a number by multiplying or dividing all the elements in one column or in one row by that number.

$$p\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} pa_1 & b_1 & c_1 \\ pa_2 & b_2 & c_2 \\ pa_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} pa_1 & pb_1 & pc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

**167.** A determinant can be split into two or more determinants. Thus,

$$\begin{vmatrix} a_1 + p_1 + q_1 & b_1 & c_1 \\ a_2 + p_2 + q_2 & b_2 & c_2 \\ a_3 + p_3 + q_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} p_1 & b_1 & c_1 \\ p_2 & b_2 & c_2 \\ p_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} q_1 & b_1 & c_1 \\ q_2 & b_2 & c_2 \\ q_3 & b_3 & c_3 \end{vmatrix}.$$

**168.** A determinant is not changed in value when the elements of one column or row are each increased or diminished by $n$ times the corresponding elements of a parallel column or row. Thus,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 \pm nb_1 & b_1 & c_1 \\ a_2 \pm nb_2 & b_2 & c_2 \\ a_3 \pm nb_3 & b_3 & c_3 \end{vmatrix}.$$

*The Solution of Equations of the First Degree by Determinants.*

**169.** The solution of

$$\left.\begin{aligned} a_1x + b_1y &= k_1 \\ a_2x + b_2y &= k_2 \end{aligned}\right\} \text{ is } \left\{\begin{aligned} x &= \begin{vmatrix} k_1 b_1 \\ k_2 b_2 \end{vmatrix} \div \begin{vmatrix} a_1 b_1 \\ a_2 b_2 \end{vmatrix} \\ y &= \begin{vmatrix} a_1 k_1 \\ a_2 k_2 \end{vmatrix} \div \begin{vmatrix} a_1 b_1 \\ a_2 b_2 \end{vmatrix} \end{aligned}\right.$$

**170.** The solution of

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= k_1 \\ a_2x + b_2y + c_2z &= k_2 \\ a_3x + b_3y + c_3z &= k_3 \end{aligned}\right\}, \text{ by putting } \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = D,$$

is $$x = \begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix} \div D; \quad y = \begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix} \div D; \quad z = \begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix} \div D.$$

The same method applies to a set of $n$ equations with $n$ unknown quantities. Observe that the denominator of the value of each unknown quantity is the determinant formed of the coefficients of all the unknown quantities, while the numerator is the same determinant with the column of the coefficients of that unknown quantity replaced by the column of the absolute terms *on the right-hand side of the equations.*

**171.** If
$$\left.\begin{array}{l} a_1x + b_1y + c_1z = 0 \\ a_2x + b_2y + c_2z = 0 \end{array}\right\},$$

then
$$x : y : z = \begin{vmatrix} b_1c_1 \\ b_2c_2 \end{vmatrix} : \begin{vmatrix} c_1a_1 \\ c_2a_2 \end{vmatrix} : \begin{vmatrix} a_1b_1 \\ a_2b_2 \end{vmatrix},$$
$$= (b_1c_2 - b_2c_1) : (c_1a_2 - c_2a_1) : (a_1b_2 - a_2b_1).$$

## Quadratic Equations.

**172.** Solution of $Ax^2 + Bx + C = 0$,
$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

**173.** Solution of $x^2 + px + q = 0$, $\quad x = -\frac{1}{2}p \pm \frac{1}{2}\sqrt{p^2 - 4q}$.

**174.** The roots are

real when $B^2 > 4AC$, or $p^2 > 4q$,

imaginary when $B^2 < 4AC$, or $p^2 < 4q$,

real and equal when $B^2 = 4AC$, or $p^2 = 4q$.

**175.** In all cases, the sum of the roots $= -p$,
and the product of the roots $= q$.

**176.** If $x_1$ and $x_2$ denote the roots, then the equation
$$x^2 - (x_1 + x_2)\,x + x_1x_2 = 0,$$
which may also be written
$$(x - x_1)(x - x_2) = 0,$$
is identical with $\quad x^2 + px + q = 0$.

**177.** To find two numbers whose sum and product are known, form a quadratic equation, putting the negative of the given sum for $p$ and the given product for $q$, and solve.

**178.** Any rational integral equation of the $n^{\text{th}}$ degree in $x$, may be written in the form
$$x^n + p_1x^{n-1} + p_2x^{n-2} + p_3x^{n-3} + \ldots + p_n = 0.$$

179. If $x_1, x_2, x_3, \ldots x_n$ be the $n$ roots of this equation, the first member is divisible by each of the factors $x - x_1$, $x - x_2$, etc., and is the product of them all, thus,

$$(x - x_1)(x - x_2)(x - x_3) \ldots (x - x_n) = 0.$$

180. Any equation the first member of which can be separated into factors in the form just given is solved by putting each factor equal to zero. For example, the equation $x^3 + 3x^2 - 4x - 12 = 0$, by separating the first member into factors, becomes

$$(x + 3)(x + 2)(x - 2) = 0,$$

the roots of which are $-3$, $-2$, and $+2$.

Again, $x^3 + 4x^2 + 4x + 3 = 0$, being reducible to the form

$$(x + 3)(x^2 + x + 1) = 0,$$

is solved by putting $x + 3 = 0$ and $x^2 + x + 1 = 0$.

Many cubic equations and equations of higher degrees are easily solved by this method, if the first member is separable into factors of a degree not higher than the second.

181. If the factors of the first member of 179 be multiplied together, the result is

$$x^n - (x_1 + x_2 + \ldots + x_n)x^{n-1} + (x_1x_2 + x_2x_3 + \ldots)x^{n-2}$$
$$- (x_1x_2x_3 + x_2x_3x_4 + \ldots)x^{n-3} + \ldots \pm x_1x_2x_3 \ldots x_n = 0,$$

an equation which is identical with

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_n = 0.$$

Therefore,

182. $$\begin{cases} -p_1 = \text{the sum of the roots,} \\ p_2 = \text{the sum of all the products of the roots taken two by two,} \\ -p_3 = \text{the sum of all the products of the roots taken three by three,} \\ \cdots\cdots\cdots\cdots \\ (-1)^k p_k = \text{the sum of all the products of the roots taken } k \text{ by } k, \\ \cdots\cdots\cdots\cdots \\ (-1)^n p_n = \text{the product of all the roots.} \end{cases}$$

## Cubic Equations.

183. To solve the general cubic equation

$$x^3 + ax^2 + bx + c = 0,$$

remove the second term by substituting for $x$ an assumed unknown, $y - \frac{1}{3}a$. The reduced equation takes the form

$$y^3 + py + q = 0.$$

184. The three roots of this last equation by Cardan's Rule are

$$y_1 = \sqrt[3]{-\tfrac{1}{2}q + \sqrt{(\tfrac{1}{2}q)^2 + (\tfrac{1}{3}p)^3}} + \sqrt[3]{-\tfrac{1}{2}q - \sqrt{(\tfrac{1}{2}q)^2 + (\tfrac{1}{3}p)^3}},$$

$$y_2 = w_1\sqrt[3]{-\tfrac{1}{2}q + \sqrt{(\tfrac{1}{2}q)^2 + (\tfrac{1}{3}p)^3}} + w_2\sqrt[3]{-\tfrac{1}{2}q - \sqrt{(\tfrac{1}{2}q)^2 + (\tfrac{1}{3}p)^3}},$$

$$y_3 = w_2\sqrt[3]{-\tfrac{1}{2}q + \sqrt{(\tfrac{1}{2}q)^2 + (\tfrac{1}{3}p)^3}} + w_1\sqrt[3]{-\tfrac{1}{2}q - \sqrt{(\tfrac{1}{2}q)^2 + (\tfrac{1}{3}p)^3}},$$

wherein $\quad w_1 = \dfrac{-1 + i\sqrt{3}}{2}, \qquad w_2 = \dfrac{-1 - i\sqrt{3}}{2}.$

When the quantity $(\frac{1}{2}q)^2 + (\frac{1}{3}p)^3$ is negative, the solution may be effected by means of circular or of hyperbolic functions in the following way:

(*i*) When $y^3 + py \pm q = 0$, $p$ and $q$ being positive, compute the value of $\varphi$ from

$$\operatorname{Sinh}\varphi = \frac{\frac{1}{2}q}{\frac{1}{3}p\sqrt{\frac{1}{3}p}}.$$

Then the roots are

185. $$\begin{cases} y_1 = \mp 2\sqrt{\tfrac{1}{3}p}\,\operatorname{Sinh}\tfrac{1}{3}\varphi, \\ y_2 = \pm\ \sqrt{\tfrac{1}{3}p}\,\operatorname{Sinh}\tfrac{1}{3}\varphi + i\sqrt{p}\,\operatorname{Cosh}\tfrac{1}{3}p, \\ y_3 = \pm\ \sqrt{\tfrac{1}{3}p}\,\operatorname{Sinh}\tfrac{1}{3}\varphi - i\sqrt{p}\,\operatorname{Cosh}\tfrac{1}{3}p. \end{cases}$$

(*ii*) When $y^3 - py \pm q = 0$, $p$ and $q$ being positive and $(\frac{1}{3}p)^3 < (\frac{1}{2}q)^2$, compute the value of $\varphi$ from

$$\operatorname{Cosh}\varphi = \frac{\frac{1}{2}q}{\frac{1}{3}p\sqrt{\frac{1}{3}p}}.$$

Then the roots are

**186.** $$\begin{cases} y_1 = \mp 2\sqrt{\tfrac{1}{3}p}\ \text{Cosh}\ \tfrac{1}{3}\varphi, \\ y_2 = \pm \sqrt{\tfrac{1}{3}p}\ \text{Cosh}\ \tfrac{1}{3}\varphi + i\sqrt{p}\ \text{Sinh}\ \tfrac{1}{3}\varphi, \\ y_3 = \pm \sqrt{\tfrac{1}{3}p}\ \text{Cosh}\ \tfrac{1}{3}\varphi - i\sqrt{p}\ \text{Sinh}\ \tfrac{1}{3}\varphi. \end{cases}$$

(*iii*) When $y^3 - py \pm q = 0$, $p$ and $q$ being positive and $(\frac{1}{3}p)^3 > (\frac{1}{2}q)^2$, compute the value of the *angle* $\varphi$ from

$$\cos\varphi = \frac{\frac{1}{2}q}{\frac{1}{3}p\sqrt{\frac{1}{3}p}}.$$

Then the roots are

**187.** $$\begin{cases} y_1 = \mp 2\sqrt{\tfrac{1}{3}p}\cos\tfrac{1}{3}\varphi, \\ y_2 = \mp 2\sqrt{\tfrac{1}{3}p}\cos(\tfrac{1}{3}\varphi + 120°), \\ y_3 = \mp 2\sqrt{\tfrac{1}{3}p}\cos(\tfrac{1}{3}\varphi + 240°). \end{cases}$$

(*iv*) When $y^3 - py \pm q = 0$, $p$ and $q$ being positive,

and $$(\tfrac{1}{3}p)^3 = (\tfrac{1}{2}q)^2,$$

**188.** the roots are $\begin{cases} y_1 = \mp 2\sqrt{\tfrac{1}{3}p}, \\ y_2 = y_3 = \pm\sqrt{\tfrac{1}{3}p}. \end{cases}$

## Series.

*Arithmetic Series.*

**189.** The $n^{\text{th}}$ term of the series

$$a,\ a+d,\ a+2d,\ a+3d, \ldots$$

is $$a + (n-1)d;$$

and the sum of $n$ terms is

**190.** $$S = \frac{n}{2}[2a + (n-1)d].$$

*Geometric Series.*

**191.** The $n^{\text{th}}$ term of the series $a, ar, ar^2, ar^3, \ldots$ is $ar^{n-1}$, and the sum of $n$ terms is

**192.** $$S = a\left(\frac{r^n - 1}{r - 1}\right).$$

If the number of terms, $n$, be infinite and the ratio, $r$, be a proper fraction, the series is convergent, and

193. $$S = \frac{a}{1-r}.$$

*Harmonic Series.*

194. The terms $a$, $b$, $c$, $d$, etc., form a harmonic series if their reciprocals $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}$, etc., form an arithmetic series, that is, when the relation subsisting between any three consecutive terms is

$$\frac{a}{c} = \frac{a-b}{b-c}.$$

195. The $n^{\text{th}}$ term in a harmonic series is

$$\frac{ab}{(n-1)a-(n-2)b}.$$

196. The arithmetic mean between $a$ and $b = \frac{a+b}{2}$.

197. The geometric mean between $a$ and $b = \sqrt{ab}$.

198. The harmonic mean between $a$ and $b = \frac{2ab}{a+b}$.

199. A series partly arithmetic and partly geometric is represented by

$$a,\ (a+d)r,\quad (a+2d)r^2,\quad (a+3d)r^3, \text{ etc.}$$

The sum of $n$ terms of this series,

$$S = \frac{a-[a+(n-1)d]r^n}{1-r} + \frac{rd(1-r^{n-1})}{(1-r)^2}.$$

200. $1+2+3+4+5+\ldots+(n-1)+n = \frac{n(n+1)}{2}$.

201. $p+(p+1)+(p+2)+\ldots+(q-1)+q$

$$= \frac{(q+p)(q-p+1)}{2}.$$

202. $2+4+6+8+\ldots+(2n-2)+2n = n(n+1)$.

203. $1+3+5+7+\ldots+(2n-3)+(2n-1) = n^2$.

204. $1^2 + 2^2 + 3^2 + 4^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{1.2.3}.$

205. $1^3 + 2^3 + 3^3 + 4^3 + \dots + n^3 = \left[\frac{n(n+1)}{2}\right]^2.$

## Binomial Series.

206.
$$(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)}{2!}x^2 \pm \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$
Convergent if $x^2 < 1$.

207.
$$(1 \pm x)^{-n} = 1 \mp nx + \frac{n(n+1)}{2!}x^2 \mp \frac{n(n+1)(n+2)}{3!}x^3 + \dots$$
Convergent if $x^2 < 1$.

208. $(a - bx)^{-1} = \frac{1}{a}\left(1 + \frac{bx}{a} + \frac{b^2x^2}{a^2} + \frac{b^3x^3}{a^3} + \dots\right)$

Convergent if $b^2x^2 < a^2$.

209. $(1 \pm x)^{-1} = 1 \mp x + x^2 \mp x^3 + x^4 \mp x^5 + \dots$

Convergent if $x^2 < 1$.

210. $(1 \pm x)^{-2} = 1 \mp 2x + 3x^2 \mp 4x^3 + 5x^4 \mp 6x^5 + \dots$

Convergent if $x^2 < 1$.

211. $(1 \pm x)^{\frac{1}{2}} = 1 \pm \frac{1}{2}x - \frac{1.1}{2.4}x^2 \pm \frac{1.1.3}{2.4.6}x^3 - \frac{1.1.3.5}{2.4.6.8}x^4 \pm \dots$

Convergent if $x^2 < 1$.

212. $(1 \pm x)^{-\frac{1}{2}} = 1 \mp \frac{1}{2}x + \frac{1.3}{2.4}x^2 \mp \frac{1.3.5}{2.4.6}x^3 + \frac{1.3.5.7}{2.4.6.8}x^4 \mp \dots$

Convergent if $x^2 < 1$.

213. $(1 \pm x)^{\frac{1}{3}} = 1 \pm \frac{1}{3}x - \frac{1.2}{3.6}x^2 \pm \frac{1.2.5}{3.6.9}x^3 - \frac{1.2.5.8}{3.6.9.12}x^4 \pm \dots$

Convergent if $x^2 < 1$.

214. $(1 \pm x)^{-\frac{1}{3}} = 1 \mp \frac{1}{3}x + \frac{1.4}{3.6}x^2 \mp \frac{1.4.7}{3.6.9}x^3 + \frac{1.4.7.10}{3.6.9.12}x^4 \mp \dots$

Convergent if $x^2 < 1$.

## Exponential and Logarithmic Series.

215. $e = 1 + \frac{1}{1} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots$

$= \text{Limit of} \left(1 + \frac{1}{m}\right)^m \text{ for } m = \infty.$

216. $e^x = 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

$[-\infty < x < +\infty.$

217. $a^x = 1 + cx + \frac{c^2x^2}{2!} + \frac{c^3x^3}{3!} + \frac{c^4x^4}{4!} + \dots$

$[-\infty < x < +\infty.$

wherein $c = \log_e a.$

218. $a^x = 1 + \frac{\log_e a}{1} x + \frac{(\log_e a)^2}{2!} x^2 + \frac{(\log_e a)^3}{3!} x^3 + \dots$

$[-\infty < x < +\infty.$

219. $\log_e (1 \pm x) = \pm x - \frac{1}{2}x^2 \pm \frac{1}{3}x^3 - \frac{1}{4}x^4 \pm \frac{1}{5}x^5 - \dots$

$[x^2 < 1.$

220. $\frac{1}{2} \log_e \frac{1+x}{1-x} = x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \dots$ $[x^2 < 1.$

221. $\frac{1}{2} \log_e \frac{x+1}{x-1} = x^{-1} + \frac{1}{3}x^{-3} + \frac{1}{5}x^{-5} + \frac{1}{7}x^{-7} + \dots$ $[x^2 > 1.$

222. $\log_e x$

$= 2\left[\frac{x-1}{x+1} + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \frac{1}{7}\left(\frac{x-1}{x+1}\right)^7 + \dots\right];$

$[0 < x < +\infty.$

223. $\log_e (a + x)$

$= \log_e a + 2\left[\frac{x}{2a+x} + \frac{1}{3}\left(\frac{x}{2a+x}\right)^3 + \frac{1}{5}\left(\frac{x}{2a+x}\right)^5 + \dots\right],$

$[0 < a < +\infty, \quad -a < x < +\infty.$

224. $\log\left(\frac{n+1}{n}\right) = \log (n+1) - \log n$

$= 2\left\{\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \dots\right\}.$

$[0 < n < +\infty.$

**225.** $\log(x+\sqrt{1+x^2})$

$$= x - \frac{1x^3}{2.3} + \frac{1.3x^5}{2.4.5} - \frac{1.3.5x^7}{2.4.6.7} + \ldots \qquad [x^2 < 1.$$

See Formulas **754** and **1036**.

**226.** $\log_e x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \ldots$ $[0 < x < 2.$

**227.** $\log_e x = \frac{x-1}{x} + \frac{1}{2}\left(\frac{x-1}{x}\right)^2 + \frac{1}{3}\left(\frac{x-1}{x}\right)^3 + \ldots$ $[x > \frac{1}{2}.$

## Interest and Annuities.

Let $r$ be the rate, that is,

$r$ = interest on one dollar for one year,
$n$ = the number of years,
$P$ = the principal,
$A$ = the amount in $n$ years.

Then,

**228.** At simple interest, $A = P(1+nr)$.

**229.** At compound interest, $A = P(1+r)^n$.

**230.** If interest be compounded $q$ times a year,

$$A = P\left(1+\frac{r}{q}\right)^{nq}.$$

If $A$ be an amount of money payable $n$ years hence, and $P$ the present worth of $A$, then

**231.** At simple interest, $P = \frac{A}{1+nr}$.

**232.** At compound interest, $P = \frac{A}{(1+r)^n}$.

**233.** Discount $= A - P$.*

* This is *true discount*, so-called to distinguish it from commercial discount, which, for commercial convenience, is based o ı a simpler rule.

234. The amount of an annuity of one dollar in $n$ years at simple interest . . . . . . . . . . $= n + \frac{n(n-1)}{2}r.$

235. Present value of such an annuity . . . . . . . . . . . $= \frac{n + \frac{1}{2}n(n-1)r}{1 + nr}.$

236. Amount at compound interest . $= \frac{(1+r)^n - 1}{(1+r) - 1}.$

237. Present value . . . . . . . . . $= \frac{1 - (1+r)^{-n}}{(1+r) - 1}.$

238. Amount when the payments of interest are made $q$ times a year . . . . . . . . . . . $= \frac{\left(1 + \frac{r}{q}\right)^{nq} - 1}{\left(1 + \frac{r}{q}\right)^{q} - 1}.$

239. Present value . . . . . . . . . $= \frac{1 - \left(1 + \frac{r}{q}\right)^{-nq}}{\left(1 + \frac{r}{q}\right)^{q} - 1}.$

240. Amount when payments of the annuity are made $m$ times a year . . . . . . . . . . . . $= \frac{(1+r)^n - 1}{m\left[(1+r)^{\frac{1}{m}} - 1\right]}.$

241. Present value. . . . . . . . . $= \frac{1 - (1+r)^{-n}}{m\left[(1+r)^{\frac{1}{m}} - 1\right]}.$

242. Amount when the interest is paid $q$ times and the annuity $m$ times a year . . . . . . . $= \frac{\left(1 + \frac{r}{q}\right)^{nq} - 1}{m\left[\left(1 + \frac{r}{q}\right)^{\frac{q}{m}} - 1\right]}.$

243. Present value. . . . . . . . . $= \frac{1 - \left(1 + \frac{r}{q}\right)^{-nq}}{m\left[\left(1 + \frac{r}{q}\right)^{\frac{q}{m}} - 1\right]}.$

## Probabilities.

If there are $a$ ways in which an event can happen, and $b$ ways in which it must fail to happen, the *chances* (or odds) in favor of the event are said to be as $a$ to $b$, and the chances (or odds) against it as $b$ to $a$.

The *probability* of an event is the ratio of the number of favorable chances to the total number of chances, both favorable and unfavorable. In the case above stated,

**244.** $\begin{cases} \dfrac{a}{a+b} = \textit{the probability} \text{ that the event may happen.} \\ \dfrac{b}{a+b} = \textit{the probability} \text{ that the event may fail to happen.} \end{cases}$

The sum of these two probabilities is 1; and since the event is *certain* either to happen or fail to happen,

**245.** $\quad$ *Certainty* $= 1$.

If $p$ be the probability of an event, the probability that that the event may fail is $1 - p$.

If $E_1$ and $E_2$ are two possible and *independent* events, and $p_1$ and $p_2$ are their respective probabilities, then

**246.** $p_1p_2 =$ the probability that *both* $E_1$ and $E_2$ may happen.

**247.** $1 - p_1p_2 =$ the probability that *not both* $E_1$ and $E_2$ may happen.

**248.** $(1-p_1)p_2 =$ the probability that $E_1$ may fail and $E_2$ happen.

**249.** $p_1(1-p_2) =$ the probability that $E_1$ may happen and $E_2$ fail.

**250.** $p_1 + p_2 - 2p_1p_2 =$ the probability that one event may happen, and the other fail.

**251.** $(1-p_1)(1-p_2) =$ the probability that both events may fail.

The value of $p$ may be determined, approximately at least, by observation of a large number of cases. Thus the experience of life assurance companies shows that out of 69,517 persons living in their fifty-first year 55,973 were living in their sixty-first year. Therefore the probability that an assured person at the age of fifty may live ten years is the ratio of these numbers, 0.805.

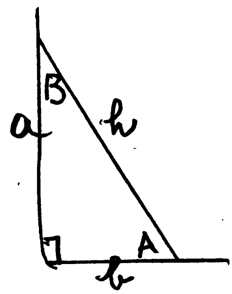
# SECTION II.

## CIRCULAR FUNCTIONS AND TRIGONOMETRY.

*Definitions and Fundamental Relations with reference to an Acute Angle.*

Denoting the legs of a right angled triangle by $a$ and $b$, the angles opposite them respectively by $A$ and $B$, and the hypotenuse by $h$, the functions of either acute angle are defined and expressed as follows:

301.
$$\begin{cases} \sin A = \dfrac{a}{h} = \cos B, \\ \cos A = \dfrac{b}{h} = \sin B, \\ \tan A = \dfrac{a}{b} = \operatorname{ctn} B, \\ \operatorname{ctn} A = \dfrac{b}{a} = \tan B, \\ \sec A = \dfrac{h}{b} = \csc B, \\ \csc A = \dfrac{h}{a} = \sec B. \end{cases}$$




The abbreviations are *sin* for *sine, cos* for *cosine, tan* for *tangent, ctn* for *cotangent, sec* for *secant*, and *csc* for *cosecant*.

From these definitions follow at once the relations,

302. $\dfrac{1}{\sin A} = \csc A,$

303. $\dfrac{1}{\csc A} = \sin A,$

304. $\dfrac{1}{\cos A} = \sec A,$

305. $\dfrac{1}{\sec A} = \cos A,$

306. $\dfrac{1}{\tan A} = \operatorname{ctn} A,$

307. $\dfrac{1}{\operatorname{ctn} A} = \tan A,$

308. $\tan A = \dfrac{\sin A}{\cos A},$

309. $\operatorname{ctn} A = \dfrac{\cos A}{\sin A}.$

And from the definitions together with the equation

$$h^2 = a^2 + b^2$$

follow the further relations,

310. $$\sin^2 A + \cos^2 A = 1,$$

311. $$\sec^2 A = 1 + \tan^2 A,$$

312. $$\csc^2 A = 1 + \text{ctn}^2 A.$$

Also, since $$B = 90^\circ - A,$$

313. $$\begin{cases} \sin\,(90^\circ - A) = \cos A, \\ \cos\,(90^\circ - A) = \sin A, \\ \tan\,(90^\circ - A) = \text{ctn}\, A, \\ \text{ctn}\,(90^\circ - A) = \tan A, \\ \sec\,(90^\circ - A) = \csc A, \\ \csc\,(90^\circ - A) = \sec A. \end{cases}$$

Therefore, if the values of all the functions of each angle from 0° to 45° are given (as in the table on page 249), the values of the functions of all angles from 90° to 45° are given also.

The functions of acute angles as above defined, when computed and tabulated, are sufficient for the solution of right triangles in all cases. They are also sufficient for the solution of an oblique triangle, if the latter be converted into the sum or the difference of two right triangles by drawing a perpendicular from a vertex to the opposite side or to the opposite side extended. For methods of solution, see pages 60–66,

*General Definitions of Angle, its Measures, and its Functions.*

314. An angle is any amount of turning in a fixed plane by which a straight line may be changed from one direction to any other direction in that plane.

If the turning amount to less than a quarter of a revolution the angle is a geometric acute angle; if to more than a

quarter and less than a half of a revolution, it is a geometric obtuse angle; if to more than a half and less than a whole revolution, it is a so-called convex angle.

The turning may amount to more than one whole revolution or to more than any number of whole revolutions however great. Moreover, the turning may be one way, positive, or the other way, negative. Therefore the general value of an angle is expressed by

**315.** $$\pm A \pm k\,360°, \qquad \text{or} \quad \pm \alpha \pm 2k\pi,$$

wherein $k$ is any integer or 0.

Angles are measured in *degrees*, *minutes*, and *seconds;* or in units of arc-measure, called *radians*. The arc-measure is the ratio of the arc to the radius, the arc being the whole arc described by any point of the turning line, and the radius the distance of that point from the centre of revolution. The arc-measure of one whole revolution is the circumference of a circle divided by its radius, or $2\pi$. The infinite range of value which an angle, $A$, or its arc-measure, $\alpha$, takes may be thus expressed,

in degrees, $\quad -\infty° < A < +\infty°.$

in radians, $\quad -\infty < \alpha < +\infty.$

The two measures of an angle are thus related,

**316.** $$\begin{cases} 1 \text{ Radian} = 57°\ 17'\ 44''.806 \\ 180° = \pi = 3.14159265 \text{ radians.} \end{cases}$$

A table for converting either kind of measure into the other is given on the next page.

As a matter of notation in the following pages, capital italic letters will, in general, indicate that the angles are to be expressed in degrees, minutes and seconds, while Greek letters or small italics will indicate that they are to be expressed in arc-measure or radians. It is, however, in many formulas a matter of indifference which notation is used.

## 317. TABLE

**(a) For finding the Length of the Arc measuring any given Angle in a Circle of which the Radius is 1.**

| Angle in Degrees | Arc in Radians | Angle in Degrees | Arc in Radians |
|---|---|---|---|
| 90° | $\frac{1}{2}\pi = 1.57079633$ | 270° | $\frac{3}{2}\pi = 4.71238898$ |
| 180° | $\pi = 3.14159265$ | 360° | $2\pi = 6.28318531$ |

| Angle | Degrees | Minutes | Seconds |
|---|---|---|---|
| 1 | 0.0174533 | 0.0002909 | 0.0000048 |
| 2 | 0.0349066 | 0.0005818 | 0.0000097 |
| 3 | 0.0523599 | 0.0008727 | 0.0000145 |
| 4 | 0.0698132 | 0.0011636 | 0.0000194 |
| 5 | 0.0872665 | 0.0014544 | 0.0000242 |
| 6 | 0.1047198 | 0.0017453 | 0.0000291 |
| 7 | 0.1221730 | 0.0020362 | 0.0000339 |
| 8 | 0.1396263 | 0.0023271 | 0.0000388 |
| 9 | 0.1570796 | 0.0026180 | 0.0000436 |

**(b) For finding the Angle measured by any given Length of Arc in a Circle of which the Radius is 1.**

| Arc | Angle | Arc | Angle |
|---|---|---|---|
| 1 | 57° 17′ 44.8″ | 4 | 229° 10′ 59.2″ |
| 2 | 114 35 29.6 | 5 | 286 28 44.0 |
| 3 | 171 53 14.4 | 6 | 343 46 28.8 |

| | Tenths | Hundredths | Thousandths | Ten-thousandths | Hundred-thousandths |
|---|---|---|---|---|---|
| 1 | 5° 43′ 46.5″ | 0° 34′ 22.6″ | 3′ 26.3″ | 0′ 20.6″ | 2.1″ |
| 2 | 11 27 33.0 | 1 8 45.3 | 6 52.5 | 0 41.3 | 4.1 |
| 3 | 17 11 19.4 | 1 43 7.9 | 10 18.8 | 1 1.9 | 6.2 |
| 4 | 22 55 5.9 | 2 17 30.6 | 13 45.1 | 1 22.5 | 8.3 |
| 5 | 28 38 52.4 | 2 51 53.2 | 17 11.3 | 1 43.1 | 10.3 |
| 6 | 34 22 38.9 | 3 26 15.9 | 20 37.6 | 2 3.8 | 12.4 |
| 7 | 40 6 25.4 | 4 0 38.5 | 24 3.9 | 2 24.4 | 14.4 |
| 8 | 45 50 11.8 | 4 35 1.2 | 27 30.1 | 2 45.0 | 16.5 |
| 9 | 51 33 58.3 | 5 9 23.8 | 30 56.4 | 3 5.6 | 18.6 |

*Functions of the General Angle Defined.*

Drawing rectangular axes, XX′ horizontal and YY′ vertical, intersecting in O, and the line OP in any required direction, let OX be the initial side and OP the terminal side of any angle whatever (defined as in 314).

It is evident that OP may fall in any one of the four quadrants, the first XOY, the second YOX′, the third X′OY′, or the fourth Y′OX.

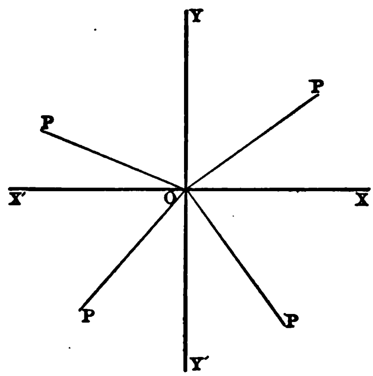



FIGURE 1.

Let the coördinates of the point P in any situation be

$x$ = the *abscissa*, or distance of P to the right or left of the vertical axis,

$y$ = the *ordinate*, or distance of P above or below the horizontal axis.

Let $r$ = the *distance* of P from O.

Then are the six functions of $A$ (any angle whatever) defined as follows:

318.
$$\begin{cases} \sin A = \dfrac{\text{ordinate}}{\text{distance}} = \dfrac{y}{r}, & \operatorname{ctn} A = \dfrac{\text{abscissa}}{\text{ordinate}} = \dfrac{x}{y}, \\ \cos A = \dfrac{\text{abscissa}}{\text{distance}} = \dfrac{x}{r}, & \sec A = \dfrac{\text{distance}}{\text{abscissa}} = \dfrac{r}{x}, \\ \tan A = \dfrac{\text{ordinate}}{\text{abscissa}} = \dfrac{y}{x}, & \csc A = \dfrac{\text{distance}}{\text{ordinate}} = \dfrac{r}{y}. \end{cases}$$

An angle is said to be an angle of the first, second, third, or fourth quadrant according as its terminal side falls in the first, second, third, or fourth quadrant.

The functions of angles of different quadrants have positive or negative values dependent on the values of the coördinates used in the definitions **318.** The *abscissa* is positive or negative according as P is to the right or left of the vertical axis; the *ordinate* is positive or negative according as P is above or below the horizontal axis; the *distance* OP is positive in all situations.

Hence the values of the functions of angles of the several quadrants are positive or negative as indicated below.

**319.**

| Quadrant | sin | cos | tan | ctn | sec | csc |
|---|---|---|---|---|---|---|
| I. | + | + | + | + | + | + |
| II. | + | − | − | − | − | + |
| III. | − | − | + | + | − | − |
| IV. | − | + | − | − | + | − |

As an angle increases from 0° to 90°, 180°, 270°, 360°, etc., its functions vary, some increasing, some decreasing, but all reaching maximum or minimum numerical values for the cardinal values of the angle above mentioned. When the function passes through the value 0 or ∞ it changes its sign, as is indicated in the following table.

**320.** **Cardinal Values.**

| Angle | Arc | Sin | Cos | Tan | Ctn | Sec | Csc |
|---|---|---|---|---|---|---|---|
| 0° | 0 | $\mp 0$ | $+1$ | $\mp 0$ | $\mp\infty$ | $+1$ | $\mp\infty$ |
| | | incr. | decr. | incr. | decr. | incr. | decr. |
| 90° | $\frac{1}{2}\pi$ | $+1$ | $\pm 0$ | $\pm\infty$ | $\pm 0$ | $\pm\infty$ | $+1$ |
| | | decr. | decr. | incr. | decr. | incr. | incr. |
| 180° | $\pi$ | $\pm 0$ | $-1$ | $\mp 0$ | $\mp\infty$ | $-1$ | $\pm\infty$ |
| | | decr. | incr. | incr. | decr. | decr. | incr. |
| 270° | $\frac{3}{2}\pi$ | $-1$ | $\mp 0$ | $\pm\infty$ | $\pm 0$ | $\mp\infty$ | $-1$ |
| | | incr. | incr. | incr. | decr. | decr. | decr. |
| 360° | $2\pi$ | $\mp 0$ | $+1$ | $\mp 0$ | $\mp\infty$ | $+1$ | $\mp\infty$ |

*Fundamental Relations Generalized.*

From the definitions **318** follow at once the relations

**321.**
$$\begin{cases} \sin A \csc A = 1, & \tan A = \dfrac{\sin A}{\cos A}, \\ \cos A \sec A = 1, & \\ \tan A \operatorname{ctn} A = 1, & \operatorname{ctn} A = \dfrac{\cos A}{\sin A}; \end{cases}$$

and from the equation $x^2 + y^2 = r^2$ follow the relations

**322.**
$$\begin{cases} \sin^2 A + \cos^2 A = 1, \\ 1 + \tan^2 A = \sec^2 A, \\ 1 + \operatorname{ctn}^2 A = \csc^2 A, \end{cases}$$

which are identical with **302–312**, as they should be; but these are applicable to angles (or arcs) of all magnitudes positive or negative, while those relate only to positive acute angles.

From **322** result six radical forms,

**323.**
$$\begin{cases} \sin A = \pm\sqrt{1 - \cos^2 A}, \\ \cos A = \pm\sqrt{1 - \sin^2 A}, \\ \tan A = \pm\sqrt{\sec^2 A - 1}, \\ \operatorname{ctn} A = \pm\sqrt{\csc^2 A - 1}, \\ \sec A = \pm\sqrt{1 + \tan^2 A}, \\ \csc A = \pm\sqrt{1 + \operatorname{ctn}^2 A}. \end{cases}$$

The interpretation of the double signs of these radicals is found in the fact that to a given value of any one function belong two angles between 0° and 360°; and the other five functions of these two angles are numerically equal each to each; but four of them have opposite algebraic signs. These four are the ones which are given by the quadratic solutions. The fifth is the reciprocal of the given one, and, like that, has the same value for the two angles.

*Anti-Functions.*

If $\sin A = x$, or $\sin \alpha = x$, then $A$ is the angle the sine of which is $x$, or $\alpha$ is the arc the sine of which is $x$, a relation usually expressed by the notation

$$A = \sin^{-1} x, \quad \text{or} \quad \alpha = \sin^{-1} x,$$

and read "$A$ (or $\alpha$) is the *anti-sine* of $x$."

Some writers use the notation arc-sin $x$ instead of $\sin^{-1} x$.

Similarly, if $\tan B = y$, $B = \tan^{-1} y$

$\sec C = z$, $C = \sec^{-1} z$.

The value of $\sin^{-1} x$ is not only $A$, as given above, but any one of the infinite series of angles included in the general expression $A \pm k\,360°$, or $\alpha \pm 2k\pi$.

Hence, $k$ being any integer, including 0,

324. $$\begin{cases} \sin^{-1} x = A \pm k\,360° = \alpha \pm 2k\pi, \\ \tan^{-1} y = B \pm k\,360° = \beta \pm 2k\pi, \\ \sec^{-1} z = C \pm k\,360° = \gamma \pm 2k\pi. \end{cases}$$

Also,

325. $$\begin{cases} \cos^{-1} x = (90° - A) \pm k\,360° = (\tfrac{1}{2}\pi - \alpha) \pm 2k\pi, \\ \operatorname{ctn}^{-1} y = (90° - B) \pm k\,360° = (\tfrac{1}{2}\pi - \beta) \pm 2k\pi, \\ \csc^{-1} z = (90° - C) \pm k\,360° = (\tfrac{1}{2}\pi - \gamma) \pm 2k\pi. \end{cases}$$

Whence

326. $$\begin{cases} \sin^{-1} x + \cos^{-1} x = 90° \pm k\,360° = \tfrac{1}{2}\pi \pm 2k\pi. \\ \tan^{-1} y + \operatorname{ctn}^{-1} y = 90° \pm k\,360° = \tfrac{1}{2}\pi \pm 2k\pi. \\ \sec^{-1} z + \csc^{-1} z = 90° \pm k\,360° = \tfrac{1}{2}\pi \pm 2k\pi. \end{cases}$$

327. *Values of Functions for Certain Angles.*

| Angle | Arc | Sin | Cos | Tan | Ctn | Sec | Csc | Chord |
|---|---|---|---|---|---|---|---|---|
| 0° | 0 | 0 | +1 | 0 | $\infty$ | +1 | $\infty$ | 0 |
| 30° | $\frac{1}{6}\pi$ | $\frac{1}{2}$ | $\frac{1}{2}\sqrt{3}$ | $\frac{1}{3}\sqrt{3}$ | $\sqrt{3}$ | $\frac{2}{3}\sqrt{3}$ | 2 | $\sqrt{2-\sqrt{3}}$ |
| 45° | $\frac{1}{4}\pi$ | $\frac{1}{2}\sqrt{2}$ | $\frac{1}{2}\sqrt{2}$ | 1 | 1 | $\sqrt{2}$ | $\sqrt{2}$ | $\sqrt{2-\sqrt{2}}$ |
| 60° | $\frac{1}{3}\pi$ | $\frac{1}{2}\sqrt{3}$ | $\frac{1}{2}$ | $\sqrt{3}$ | $\frac{1}{3}\sqrt{3}$ | 2 | $\frac{2}{3}\sqrt{3}$ | 1 |
| 90° | $\frac{1}{2}\pi$ | +1 | 0 | $\infty$ | 0 | $\infty$ | +1 | $\sqrt{2}$ |
| 120° | $\frac{2}{3}\pi$ | $\frac{1}{2}\sqrt{3}$ | $-\frac{1}{2}$ | $-\sqrt{3}$ | $-\frac{1}{3}\sqrt{3}$ | $-2$ | $\frac{2}{3}\sqrt{3}$ | $\sqrt{3}$ |
| 135° | $\frac{3}{4}\pi$ | $\frac{1}{2}\sqrt{2}$ | $-\frac{1}{2}\sqrt{2}$ | $-1$ | $-1$ | $-\sqrt{2}$ | $\sqrt{2}$ | $\sqrt{2+\sqrt{2}}$ |
| 150° | $\frac{5}{6}\pi$ | $\frac{1}{2}$ | $-\frac{1}{2}\sqrt{3}$ | $-\frac{1}{3}\sqrt{3}$ | $-\sqrt{3}$ | $-\frac{2}{3}\sqrt{3}$ | 2 | $\sqrt{2+\sqrt{3}}$ |
| 180° | $\pi$ | 0 | $-1$ | 0 | $\infty$ | $-1$ | $\infty$ | 2 |
| 210° | $\frac{7}{6}\pi$ | $-\frac{1}{2}$ | $-\frac{1}{2}\sqrt{3}$ | $\frac{1}{3}\sqrt{3}$ | $\sqrt{3}$ | $-\frac{2}{3}\sqrt{3}$ | $-2$ | $\sqrt{2+\sqrt{3}}$ |
| 225° | $\frac{5}{4}\pi$ | $-\frac{1}{2}\sqrt{2}$ | $-\frac{1}{2}\sqrt{2}$ | 1 | 1 | $-\sqrt{2}$ | $-\sqrt{2}$ | $\sqrt{2+\sqrt{2}}$ |
| 240° | $\frac{4}{3}\pi$ | $-\frac{1}{2}\sqrt{3}$ | $-\frac{1}{2}$ | $\sqrt{3}$ | $\frac{1}{3}\sqrt{3}$ | $-2$ | $-\frac{2}{3}\sqrt{3}$ | $\sqrt{3}$ |
| 270° | $\frac{3}{2}\pi$ | $-1$ | 0 | $\infty$ | 0 | $\infty$ | $-1$ | $\sqrt{2}$ |
| 300° | $\frac{5}{3}\pi$ | $-\frac{1}{2}\sqrt{3}$ | $\frac{1}{2}$ | $-\sqrt{3}$ | $-\frac{1}{3}\sqrt{3}$ | 2 | $-\frac{2}{3}\sqrt{3}$ | 1 |
| 315° | $\frac{7}{4}\pi$ | $-\frac{1}{2}\sqrt{2}$ | $\frac{1}{2}\sqrt{2}$ | $-1$ | $-1$ | $\sqrt{2}$ | $-\sqrt{2}$ | $\sqrt{2-\sqrt{2}}$ |
| 330° | $\frac{11}{6}\pi$ | $-\frac{1}{2}$ | $\frac{1}{2}\sqrt{3}$ | $-\frac{1}{3}\sqrt{3}$ | $-\sqrt{3}$ | $\frac{2}{3}\sqrt{3}$ | $-2$ | $\sqrt{2-\sqrt{3}}$ |
| 360° | $2\pi$ | 0 | +1 | 0 | $\infty$ | +1 | $\infty$ | 0 |

$\sqrt{2}=1.4142136$ $\qquad$ $\sqrt{3}=1.7320508$

$\frac{1}{2}\sqrt{2}=0.7071068$ $\qquad$ $\frac{1}{3}\sqrt{3}=0.5773503$

$\frac{2}{3}\sqrt{3}=1.1547005$

$\sqrt{2-\sqrt{2}}=0.7653669$ $\qquad$ $\sqrt{2-\sqrt{3}}=0.5176381$

$\sqrt{2+\sqrt{2}}=1.8477587$ $\qquad$ $\sqrt{2+\sqrt{3}}=1.9318516$

*Formulas expressing each function in terms of each of the others.*

**328.** $\sin A = \pm\sqrt{1-\cos^2 A} = \dfrac{\tan A}{\pm\sqrt{1+\tan^2 A}}$

$$= \frac{1}{\pm\sqrt{1+\operatorname{ctn}^2 A}} = \frac{\pm\sqrt{\sec^2 A - 1}}{\sec A} = \frac{1}{\csc A}.$$

**329.** $\cos A = \pm\sqrt{1-\sin^2 A} = \dfrac{1}{\pm\sqrt{1+\tan^2 A}}$

$$= \frac{\operatorname{ctn} A}{\pm\sqrt{1+\operatorname{ctn}^2 A}} = \frac{1}{\sec A} = \frac{\pm\sqrt{\csc^2 A - 1}}{\csc A}.$$

**330.** $\tan A = \dfrac{\sin A}{\pm\sqrt{1-\sin^2 A}} = \dfrac{\pm\sqrt{1-\cos^2 A}}{\cos A} = \dfrac{1}{\operatorname{ctn} A},$

$$= \pm\sqrt{\sec^2 A - 1} = \frac{1}{\pm\sqrt{\csc^2 A - 1}}.$$

**331.** $\operatorname{ctn} A = \dfrac{\pm\sqrt{1-\sin^2 A}}{\sin A} = \dfrac{\cos A}{\pm\sqrt{1-\cos^2 A}} = \dfrac{1}{\tan A},$

$$= \frac{1}{\pm\sqrt{\sec^2 A - 1}} = \pm\sqrt{\csc^2 A - 1}.$$

**332.** $\sec A = \dfrac{1}{\pm\sqrt{1-\sin^2 A}} = \dfrac{1}{\cos A} = \pm\sqrt{1+\tan^2 A},$

$$= \frac{\pm\sqrt{1+\operatorname{ctn}^2 A}}{\operatorname{ctn} A} = \frac{\csc A}{\pm\sqrt{\csc^2 A - 1}}.$$

**333.** $\csc A = \dfrac{1}{\sin A} = \dfrac{1}{\pm\sqrt{1-\cos^2 A}} = \dfrac{\pm\sqrt{1+\tan^2 A}}{\tan A},$

$$= \pm\sqrt{1+\operatorname{ctn}^2 A} = \frac{\sec A}{\pm\sqrt{\sec^2 A - 1}}.$$

*Positive and Negative Lines.*

If the distance from a point A to any other point B on a straight line be reckoned as positive, then the distance from B to A must be reckoned as negative; so that it is always true that

334. $$AB + BA = 0.$$

Let three points A, B, and C be arranged in any order on a straight line. Then the algebraic sum of the distances AB and BC is always AC, that is,

335. $$AB + BC = AC,$$

which by adding CA to each member becomes

336. $$AB + BC + CA = 0.$$

The same principle applies to any number of points, arranged in any order whatever on a straight line, and their distances, that is

337. $$AB + BC + CD + \dots + MN + NA = 0.$$

*Projections.*

The projections of a line AB upon the axes of $x$ and $y$ are

338. $$\begin{cases} AB \cos A = \text{projection on the axis of } x, \\ AB \sin A = \text{projection on the axis of } y, \end{cases}$$

wherein $A$ denotes the angle between the positive direction of the axis of $x$ and the positive direction of the projected line.

The sum of the projections of the sides of any closed polygon, taken in order around the polygon, upon any chosen line is equal to 0.

In the case of a triangle ABC placed anyhow in the plane of the axes OX and OY, if A′, B′, C′ be the projections of the points A, B, C on the axis of $x$ and A″, B″, C″, the projections of the same points on the axis of $y$, then, whatever the order in which the projections fall on either axis,

339. $$\begin{cases} A'B' + B'C' + C'A' = 0. \\ A''B'' + B''C'' + C''A'' = 0. \end{cases}$$

If the axes be rectangular and ABC a right-angled triangle, these equations give the formulas for the sine and the cosine of the sum and of the difference of two angles.

### *Positive and Negative Angles.*

If the angle AOB be reckoned as positive, then the angle BOA must be reckoned as negative; so that it is always true that

**340.** $\text{AOB} + \text{BOA} = 0$, or $= \pm k\, 360°$, or $= \pm 2k\pi$.

Also, whatever the order of the lines radiating from O,

**341.** $\text{AOB} + \text{BOC} = \text{AOC}$, or $= \text{AOC} \pm k\, 360°$,
or $= \text{AOC} \pm 2k\pi$.

**342.** $\text{AOB} + \text{BOC} + \text{COA} = 0$, or $= \pm k\, 360°$, or $= 2k\pi$.

**343.** $\text{AOB} + \text{BOC} + \text{COD} + \ldots + \text{MON} + \text{NOA} = 0$,
or $= \pm k\, 360°$, or $= \pm 2k\pi$.

### *Functions of the Sum and of the Difference of Two Angles.*

**344.** $\sin (A + B) = \sin A \cos B + \cos A \sin B.$

**345.** $\sin (A - B) = \sin A \cos B - \cos A \sin B.$

**346.** $\cos (A + B) = \cos A \cos B - \sin A \sin B.$

**347.** $\cos (A - B) = \cos A \cos B + \sin A \sin B.$

**348.** $$\tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

**349.** $$\tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}.$$

**350.** $$\text{ctn} (A + B) = \frac{\text{ctn}\, B \text{ ctn}\, A - 1}{\text{ctn}\, B + \text{ctn}\, A}.$$

**351.** $$\text{ctn} (A - B) = \frac{\text{ctn}\, B \text{ ctn}\, A + 1}{\text{ctn}\, B - \text{ctn}\, A}.$$

*Functions of the Sum of Three Angles.*

352. $\sin(A+B+C)$
$$= -\sin A \sin B \sin C + \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C.$$

353. $\cos(A+B+C)$
$$= \cos A \cos B \cos C - \cos A \sin B \sin C - \sin A \cos B \sin C - \sin A \sin B \cos C.$$

*Functions of a Negative Angle.*

354. $$\begin{cases} \sin(-A) = -\sin A, & \text{ctn}(-A) = -\text{ctn}\, A, \\ \cos(-A) = \cos A, & \sec(-A) = \sec A, \\ \tan(-A) = -\tan A, & \csc(-A) = -\csc A. \end{cases}$$

*Functions of $A + 90°$.*

355. $$\begin{cases} \sin(A+90°) = \cos A, \\ \cos(A+90°) = -\sin A, \\ \tan(A+90°) = -\text{ctn}\, A, \\ \text{etc.} \end{cases}$$

*Functions of $A - 90°$.*

356. $$\begin{cases} \sin(A-90°) = -\cos A, \\ \cos(A-90°) = \sin A, \\ \tan(A-90°) = -\text{ctn}\, A, \\ \text{etc.} \end{cases}$$

*Functions of $90° - A$.*

357. $$\begin{cases} \sin(90°-A) = \cos A, \\ \cos(90°-A) = \sin A, \\ \tan(90°-A) = \text{ctn}\, A, \\ \text{etc.} \end{cases}$$

*Functions of $A + 180°$.*

358. $$\begin{cases} \sin(A+180°) = -\sin A, \\ \cos(A+180°) = -\cos A, \\ \tan(A+180°) = \tan A, \\ \text{etc.} \end{cases}$$

*Functions of* $A - 180°$.

359. $$\begin{cases} \sin (A - 180°) = -\sin A, \\ \cos (A - 180°) = -\cos A, \\ \tan (A - 180°) = \tan A, \\ \text{etc.} \end{cases}$$

*Functions of* $180° - A$.

360. $$\begin{cases} \sin (180° - A) = \sin A, \\ \cos (180° - A) = -\cos A, \\ \tan (180° - A) = -\tan A, \\ \text{etc.} \end{cases}$$

*Functions of* $A + 270°$.

361. $$\begin{cases} \sin (A + 270°) = -\cos A, \\ \cos (A + 270°) = \sin A, \\ \tan (A + 270°) = -\operatorname{ctn} A, \\ \text{etc.} \end{cases}$$

*Functions of* $A - 270°$.

362. $$\begin{cases} \sin (A - 270°) = \cos A, \\ \cos (A - 270°) = -\sin A, \\ \tan (A - 270°) = -\operatorname{ctn} A, \\ \text{etc.} \end{cases}$$

*Functions of* $270° - A$.

363. $$\begin{cases} \sin (270° - A) = -\cos A, \\ \cos (270° - A) = -\sin A, \\ \tan (270° - A) = \operatorname{ctn} A, \\ \text{etc.} \end{cases}$$

*Functions of* $A \pm 360°$.

364. $$\begin{cases} \sin (A \pm 360°) = \sin A, \\ \cos (A \pm 360°) = \cos A, \\ \tan (A \pm 360°) = \tan A, \\ \text{etc.} \end{cases}$$

*Functions of* $360° - A$.

365. $\begin{cases} \sin(360° - A) = -\sin A, \\ \cos(360° - A) = \cos A, \\ \tan(360° - A) = -\tan A, \\ \text{etc.} \end{cases}$

*Solution of equations* $\sin A = a$, $\cos A = a$, and $\tan A = a$.

If $A$ is to be found from a given value $a$ of its sine, that is, if the equation $\sin A = a$ is to be solved for $A$, *all* the values of $A$ are given by the formula

366. $$\sin^{-1} a = k\, 180° + (-1)^k A,$$

wherein $k$ is 0 or any integer.

In the same way, all the values of $A$ obtainable from the equation $\cos A = a$ are given by

367. $$\cos^{-1} a = k\, 360° \pm A.$$

And all values of $A$ obtainable from $\tan A = a$ are given by

368. $$\tan^{-1} a = k\, 180° + A.$$

*Sums and Products of Functions.*

369. $\sin A \cos B = \frac{1}{2}\sin(A+B) + \frac{1}{2}\sin(A-B)$.

370. $\cos A \sin B = \frac{1}{2}\sin(A+B) - \frac{1}{2}\sin(A-B)$.

371. $\sin A \sin B = \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)$.

372. $\cos A \cos B = \frac{1}{2}\cos(A-B) + \frac{1}{2}\cos(A+B)$.

373. $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B,$
$\qquad = \cos^2 B - \cos^2 A.$

374. $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B,$
$\qquad = \cos^2 B - \sin^2 A.$

375. $\sin A + \sin B = 2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$.

376. $\sin A - \sin B = 2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$.

377. $\cos A + \cos B = 2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$.

378. $\cos A - \cos B = -2\sin\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$.

379. $$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}.$$

380. $$\frac{\cos A - \cos B}{\cos A + \cos B} = -\tan \tfrac{1}{2}(A+B) \tan \tfrac{1}{2}(A-B).$$

381. $$\frac{\sin A \pm \sin B}{\cos A + \cos B} = \tan \tfrac{1}{2}(A \pm B).$$

382. $$\frac{\sin A \mp \sin B}{\cos A - \cos B} = -\operatorname{ctn} \tfrac{1}{2}(A \pm B).$$

383. $$\frac{\sin (A \mp B)}{\sin A \pm \sin B} = \frac{\sin \frac{1}{2}(A \mp B)}{\sin \frac{1}{2}(A \pm B)}.$$

384. $$\frac{\sin (A \mp B)}{\sin A \mp \sin B} = \frac{\cos \frac{1}{2}(A \mp B)}{\cos \frac{1}{2}(A \pm B)}.$$

385. $$\tan A \pm \tan B = \frac{\sin (A \pm B)}{\cos A \cos B}.$$

386. $$\operatorname{ctn} B \pm \operatorname{ctn} A = \frac{\sin (A \pm B)}{\sin A \sin B}.$$

387. $$\operatorname{ctn} B \pm \tan A = \frac{\cos (A \mp B)}{\cos A \sin B}.$$

388. $$\frac{\sin (A \pm B)}{\sin (A \mp B)} = \frac{\tan A \pm \tan B}{\tan A \mp \tan B} = \frac{\operatorname{ctn} B \pm \operatorname{ctn} A}{\operatorname{ctn} B \mp \operatorname{ctn} A}.$$

389. $$\frac{\cos (A \pm B)}{\cos (A \mp B)} = \frac{1 \mp \tan A \tan B}{1 \pm \tan A \tan B} = \frac{\operatorname{ctn} B \mp \tan A}{\operatorname{ctn} B \pm \tan A}.$$

390. $$\frac{\cos (A \mp B)}{\sin (A \pm B)} = \frac{\operatorname{ctn} B \pm \tan A}{\operatorname{ctn} B \tan A \pm 1}.$$

391. $$\cos^2 A + \cos^2 B - 2 \cos A \cos B \cos (A+B) = \sin^2 (A+B).$$

392. $$\frac{\tan A + \tan B}{\operatorname{ctn} A + \operatorname{ctn} B} = \tan A \tan B.$$

393. $$\sec^2 A + \csc^2 A = \sec^2 A \csc^2 A.$$

*Functions of Multiple Angles.*

394. $\sin kA = 2\sin(k-1)A\cos A - \sin(k-2)A,$
$= 2\cos(k-1)A\sin A + \sin(k-2)A.$

395. $\cos kA = 2\cos(k-1)A\cos A - \cos(k-2)A,$
$= -2\sin(k-1)A\sin A + \cos(k-2)A.$

396. $\tan kA = \dfrac{\tan(k-1)A + \tan A}{1 - \tan(k-1)A\tan A}.$

397. $\sin 2A = 2\sin A\cos A.$
$\sin 3A = 3\sin A - 4\sin^3 A.$
$\sin 4A = 4\sin A\cos A - 8\sin^3 A\cos A.$
$\sin 5A = 5\sin A - 20\sin^3 A + 16\sin^5 A.$
$\sin 6A = 6\sin A\cos A - 32\sin^3 A\cos A$
$+ 32\sin^5 A\cos A.$

398. $\cos 2A = 2\cos^2 A - 1.$
$\cos 3A = 4\cos^3 A - 3\cos A.$
$\cos 4A = 8\cos^4 A - 8\cos^2 A + 1.$
$\cos 5A = 16\cos^5 A - 20\cos^3 A + 5\cos A.$
$\cos 6A = 32\cos^6 A - 48\cos^4 A + 18\cos^2 A - 1.$

399. $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}.$

400. $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}.$

401. $\operatorname{ctn} 2A = \dfrac{\operatorname{ctn}^2 A - 1}{2\operatorname{ctn} A} = \dfrac{1 - \tan^2 A}{2\tan A} = \dfrac{\operatorname{ctn} A - \tan A}{2}.$

402. $\sec 2A = \dfrac{\sec^2 A}{1 - \tan^2 A} = \dfrac{\operatorname{ctn} A + \tan A}{\operatorname{ctn} A - \tan A}.$

403. $\csc 2A = \frac{1}{2}\sec A\csc A = \frac{1}{2}(\tan A + \operatorname{ctn} A).$

404. $1 + \sin 2A = (\sin A + \cos A)^2.$

405. $1 - \sin 2A = (\sin A - \cos A)^2.$

406. $1 + \cos 2A = 2\cos^2 A.$

407. $1 - \cos 2A = 2\sin^2 A.$

408. $\csc 2A + \operatorname{ctn} 2A = \operatorname{ctn} A.$

*Functions of Half an Angle.*

409. $1 = \sin^2 \tfrac{1}{2}A + \cos^2 \tfrac{1}{2}A.$

410. $\sin A = 2 \sin \tfrac{1}{2}A \cos \tfrac{1}{2}A.$

411. $\cos A = \cos^2 \tfrac{1}{2}A - \sin^2 \tfrac{1}{2}A.$

412. $1 + \sin A = (\sin \tfrac{1}{2}A + \cos \tfrac{1}{2}A)^2.$

413. $1 - \sin A = (\sin \tfrac{1}{2}A - \cos \tfrac{1}{2}A)^2.$

414. $1 + \cos A = 2 \cos^2 \tfrac{1}{2}A.$

415. $1 - \cos A = 2 \sin^2 \tfrac{1}{2}A.$

416. $$\sin \tfrac{1}{2}A = \sqrt{\tfrac{1}{2}(1 - \cos A)},$$
$$= \tfrac{1}{2}\sqrt{1 + \sin A} - \tfrac{1}{2}\sqrt{1 - \sin A}.$$

417. $$\cos \tfrac{1}{2}A = \sqrt{\tfrac{1}{2}(1 + \cos A)},$$
$$= \tfrac{1}{2}\sqrt{1 + \sin A} + \tfrac{1}{2}\sqrt{1 - \sin A}.$$

418. $$\tan \tfrac{1}{2}A = \sqrt{\frac{1 - \cos A}{1 + \cos A}} = \frac{1 - \cos A}{\sin A} = \frac{\sin A}{1 + \cos A},$$
$$= \frac{1 + \sin A - \cos A}{1 + \sin A + \cos A}.$$

419. $$\operatorname{ctn} \tfrac{1}{2}A = \sqrt{\frac{1 + \cos A}{1 - \cos A}} = \frac{1 + \cos A}{\sin A} = \frac{\sin A}{1 - \cos A},$$
$$= \frac{1 + \sin A + \cos A}{1 + \sin A - \cos A}.$$

420. $$\sec \tfrac{1}{2}A = \sqrt{\frac{2 \sec A}{\sec A + 1}}.$$

421. $$\csc \tfrac{1}{2}A = \sqrt{\frac{2 \sec A}{\sec A - 1}}.$$

422. $1 + \sin A = 2 \sin^2 (45° + \tfrac{1}{2}A) = 2 \cos^2 (45° - \tfrac{1}{2}A).$

423. $1 - \sin A = 2 \sin^2 (45° - \tfrac{1}{2}A) = 2 \cos^2 (45° + \tfrac{1}{2}A).$

424. $$\tan (45° \pm A) = \operatorname{ctn} (45° \mp A) = \frac{1 \pm \tan A}{1 \mp \tan A},$$
$$= \sqrt{\frac{1 \pm \sin 2A}{1 \mp \sin 2A}} = \frac{\cos A \pm \sin A}{\cos A \mp \sin A}.$$

425. $\tan (45° \pm \frac{1}{2}A) = \text{ctn} (45° \mp \frac{1}{2}A) = \sqrt{\frac{1 \pm \sin A}{1 \mp \sin A}}$,

$$= \frac{1 \pm \sin A}{\cos A} = \sec A \pm \tan A$$

$$= \frac{\cos A}{1 \mp \sin A}.$$

426. $\tan (A - 45°) = \frac{\tan A - 1}{\tan A + 1}$.

427. $\sin (45° + A) = \cos (45° - A) = \frac{\sin A + \cos A}{\sqrt{2}}$.

428. $\cos (45° + A) = \sin (45° - A) = \frac{\cos A - \sin A}{\sqrt{2}}$.

429. $\tan (45° + A) + \tan (45° - A) = 2 \sec 2A$.

430. $\tan (45° + A) - \tan (45° - A) = 2 \tan 2A$.

431. $\tan (45° + A) \tan (45° - A) = 1$.

432. $\sin (30° + A) + \sin (30° - A) = \cos A$.

433. $\sin (30° + A) - \sin (30° - A) = \sqrt{3} \sin A$.

434. $\cos (30° + A) + \cos (30° - A) = \sqrt{3} \cos A$.

435. $\cos (30° + A) - \cos (30° - A) = - \sin A$.

*Expressions Equivalent to* sin *A*.

436. $\sin A = \sqrt{1 - \cos^2 A} = \sqrt{(1 + \cos A)(1 - \cos A)}$,

$$= \cos A \tan A = \frac{\cos A}{\text{ctn} A} = \frac{\tan A}{\sec A} = \frac{1}{\csc A},$$

$$= \frac{\tan A}{\sqrt{1 + \tan^2 A}} = \frac{1}{\sqrt{1 + \text{ctn}^2 A}} = \frac{\sqrt{\sec^2 A - 1}}{\sec A},$$

$$= \sqrt{\tfrac{1}{2}(1 - \cos 2A)} = 2 \sin \tfrac{1}{2}A \cos \tfrac{1}{2}A,$$

$$= \frac{2 \tan \frac{1}{2}A}{1 + \tan^2 \frac{1}{2}A} = \frac{1}{\text{ctn} \frac{1}{2}A - \text{ctn} A},$$

$$= \frac{1}{\tan \frac{1}{2}A + \text{ctn} A} = \frac{2}{\tan \frac{1}{2}A + \text{ctn} \frac{1}{2}A},$$

$$= 2 \sin^2 (45° + \tfrac{1}{2}A) - 1,$$

$$= 1 - 2 \sin^2 (45° - \tfrac{1}{2}A) = \frac{1 - \tan^2 (45° - \frac{1}{2}A)}{1 + \tan^2 (45° - \frac{1}{2}A)}.$$

*Expressions Equivalent to* cos *A*.

437. $\cos A = \sqrt{1-\sin^2 A} = \sqrt{(1+\sin A)(1-\sin A)},$

$$= \sin A \operatorname{ctn} A = \frac{\sin A}{\tan A} = \frac{\operatorname{ctn} A}{\csc A} = \frac{1}{\sec A},$$

$$= \frac{\operatorname{ctn} A}{\sqrt{1+\operatorname{ctn}^2 A}} = \frac{1}{\sqrt{1+\tan^2 A}} = \frac{\sqrt{\csc^2 A - 1}}{\csc A},$$

$$= \sqrt{\tfrac{1}{2}(1+\cos 2A)} = \frac{\sin 2A}{2\sin A},$$

$$= \cos^2 \tfrac{1}{2}A - \sin^2 \tfrac{1}{2}A,$$

$$= 1 - 2\sin^2 \tfrac{1}{2}A = 2\cos^2 \tfrac{1}{2}A - 1 = \frac{1-\tan^2 \frac{1}{2}A}{1+\tan^2 \frac{1}{2}A},$$

$$= \frac{\operatorname{ctn}^2 \frac{1}{2}A - 1}{\operatorname{ctn}^2 \frac{1}{2}A + 1} = \frac{\operatorname{ctn} \frac{1}{2}A - \tan \frac{1}{2}A}{\operatorname{ctn} \frac{1}{2}A + \tan \frac{1}{2}A},$$

$$= \frac{1}{\tan A \operatorname{ctn} \frac{1}{2} A - 1} = \frac{1}{1 + \tan A \tan \frac{1}{2}A},$$

$$= \frac{2}{\tan(45^\circ + \frac{1}{2}A) + \operatorname{ctn}(45^\circ + \frac{1}{2}A)},$$

$$= 2\cos(45^\circ + \tfrac{1}{2}A)\cos(45^\circ - \tfrac{1}{2}A),$$

$$= \cos^4 \tfrac{1}{2}A - \sin^4 \tfrac{1}{2}A.$$

*Expressions Equivalent to* tan *A*.

438. $\tan A = \frac{\sin A}{\cos A} = \frac{1}{\operatorname{ctn} A} = \sqrt{\sec^2 A - 1},$

$$= \frac{1}{\sqrt{\csc^2 A - 1}} = \frac{\sin A}{\sqrt{1-\sin^2 A}} = \frac{\sqrt{1-\cos^2 A}}{\cos A},$$

$$= \sqrt{\frac{1-\cos 2A}{1+\cos 2A}} = \frac{\sin 2A}{1+\cos 2A} = \frac{1-\cos 2A}{\sin 2A},$$

$$= \csc 2A - \operatorname{ctn} 2A = \operatorname{ctn} A - 2\operatorname{ctn} 2A,$$

$$= \frac{2\tan \frac{1}{2}A}{1-\tan^2 \frac{1}{2}A} = \frac{2\operatorname{ctn} \frac{1}{2}A}{\operatorname{ctn}^2 \frac{1}{2}A - 1},$$

$$= \frac{2}{\operatorname{ctn} \frac{1}{2}A - \tan \frac{1}{2}A},$$

$$= \frac{\tan(45^\circ + \frac{1}{2}A) - \tan(45^\circ - \frac{1}{2}A)}{2},$$

$$= \frac{\tan(45^\circ + A) - 1}{\tan(45^\circ + A) + 1} = \frac{1 - \tan(45^\circ - A)}{1 + \tan(45^\circ - A)}.$$

Expressions equivalent to ctn $A$, sec $A$, and csc $A$ are the reciprocals of those above given for tan $A$, cos $A$, and sin $A$, respectively,

*Functions of Periodic Values of the Arc or Angle.*

In the following equations, $k$ is any integer positive, negative or 0.

439. $$\begin{cases} \sin k\pi = 0, & \sin \frac{2k+1}{2}\pi = (-1)^k, \\ \cos k\pi = (-1)^k, & \cos \frac{2k+1}{2}\pi = 0, \\ \tan k\pi = 0, & \tan \frac{2k+1}{2}\pi = \infty. \end{cases}$$

440. $$\sin\varphi = \pm \sin(2k\pi \pm \varphi) = \mp \sin[(2k+1)\pi \pm \varphi],$$
$$= \mp \cos\left(\frac{4k+1}{2}\pi \pm \varphi\right) = \pm \cos\left(\frac{4k-1}{2}\pi \pm \varphi\right).$$

441. $\csc\varphi = \pm \csc(2k\pi \pm \varphi) =$ etc.

442. $$\cos\varphi = \cos(2k\pi \pm \varphi) = -\cos[(2k+1)\pi \pm \varphi],$$
$$= \sin\left(\frac{4k+1}{2}\pi \pm \varphi\right) = -\sin\left(\frac{4k-1}{2}\pi \pm \varphi\right).$$

443. $\sec\varphi = \sec(2k\pi \pm \varphi) =$ etc.

444. $$\tan\varphi = \pm \tan(k\pi \pm \varphi) = \mp \operatorname{ctn}\left(\frac{2k+1}{2}\pi \pm \varphi\right).$$

445. $\operatorname{ctn}\varphi = \pm \operatorname{ctn}(k\pi \pm \varphi) =$ etc.

The formulas **440**, **442**, and **444** give the only solutions of the equations.

$$\sin\varphi = \pm\sin\alpha, \qquad \tan\varphi = \pm\tan\alpha,$$
$$\sin\varphi = \pm\cos\alpha, \qquad \tan\varphi = \pm\operatorname{ctn}\alpha,$$

and of the equivalent equations.

$$\csc\varphi = \pm\csc\alpha, \qquad \operatorname{ctn}\varphi = \pm\operatorname{ctn}\alpha,$$
$$\csc\varphi = \pm\sec\alpha, \qquad \operatorname{ctn}\varphi = \pm\tan\alpha.$$

If *any two* of the six elementary functions (not being reciprocals of each other) have equal values for $\varphi$ and $\alpha$ the only solution is

$$\varphi = 2k\pi + \alpha.$$

*Inverse Circular Functions.*

**446.** $\sin^{-1} x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$,

$$= \operatorname{ctn}^{-1}\frac{1}{x}\sqrt{1-x^2} = \sec^{-1}\frac{1}{\sqrt{1-x^2}} = \csc^{-1}\frac{1}{x},$$

$$= 2\sin^{-1}\sqrt{\tfrac{1}{2}(1-\sqrt{1-x^2})} = \tfrac{1}{2}\sin^{-1}(2x\sqrt{1-x^2}),$$

$$= 2\tan^{-1}\frac{1-\sqrt{1-x^2}}{x} = \tfrac{1}{2}\tan^{-1}\frac{2x\sqrt{1-x^2}}{1-2x^2},$$

$$= \tfrac{1}{2}\pi - \cos^{-1} x = \tfrac{1}{2}\pi - \sin^{-1}\sqrt{1-x^2}$$

$$= -\sin^{-1}(-x),$$

$$= \tfrac{1}{4}\pi + \tfrac{1}{2}\sin^{-1}(2x^2-1) = \tfrac{1}{2}\cos^{-1}(1-2x^2).$$

**447.** $\cos^{-1} x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\dfrac{1}{x}\sqrt{1-x^2}$

$$= \operatorname{ctn}^{-1}\frac{x}{\sqrt{1-x^2}} = \sec^{-1}\frac{1}{x},$$

$$= \csc^{-1}\frac{1}{\sqrt{1-x^2}}, = 2\cos^{-1}\sqrt{\tfrac{1}{2}(1+x)},$$

$$= \tfrac{1}{2}\cos^{-1}(2x^2-1) = 2\tan^{-1}\sqrt{\frac{1-x}{1+x}}$$

$$= 2\operatorname{ctn}^{-1}\sqrt{\frac{1+x}{1-x}},$$

$$= \tfrac{1}{2}\tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{2x^2-1}\right),$$

$$= \tfrac{1}{2}\pi - \sin^{-1} x = \pi - \cos^{-1}(-x),$$

$$= \tfrac{1}{2}\pi - \cos^{-1}\sqrt{1-x^2}.$$

448. $\tan^{-1} x = \sin^{-1}\dfrac{x}{\sqrt{1+x^2}} = \cos^{-1}\dfrac{1}{\sqrt{1+x^2}} = \text{ctn}^{-1}\dfrac{1}{x},$

$$= \sec^{-1}\sqrt{1+x^2} = \csc^{-1}\frac{1}{x}\sqrt{1+x^2}$$

$$= \tfrac{1}{2}\pi - \tan^{-1}\frac{1}{x} = -\tan^{-1}(-x),$$

$$= \tfrac{1}{2}\tan^{-1}\frac{2x}{1-x^2} = \tfrac{1}{2}\pi - \text{ctn}^{-1} x,$$

$$= \tfrac{1}{2}\sin^{-1}\frac{2x}{1+x^2} = \tfrac{1}{2}\cos^{-1}\frac{1-x^2}{1+x^2},$$

$$= 2\cos^{-1}\sqrt{\frac{1+\sqrt{1+x^2}}{2\sqrt{1+x^2}}}$$

$$= 2\sin^{-1}\sqrt{\frac{-1+\sqrt{1+x^2}}{2\sqrt{1+x^2}}}.$$

$$= 2\tan^{-1}\left(\frac{-1+\sqrt{1+x^2}}{x}\right)$$

449. $\sin^{-1} x + \sin^{-1} y$

$$= \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2}),$$
$$= \pi - \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2}).$$

450. $\sin^{-1} x - \sin^{-1} y$

$$= \sin^{-1}(x\sqrt{1-y^2} - y\sqrt{1-x^2}).$$

451. $\cos^{-1} x \pm \cos^{-1} y$

$$= \cos^{-1}(xy \mp \sqrt{(1-x^2)(1-y^2)}).$$

[449–451: $x^2 + y^2 \leqq 1.$]

452. $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\dfrac{x+y}{1-xy},$ [$xy \leqq 1.$

453. $\tan^{-1} x + \tan^{-1} y = \pi - \tan^{-1}\dfrac{x+y}{xy-1},$ [$xy \geqq 1.$

454. $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\dfrac{x-y}{1+xy}.$

*Relations of Circular, Exponential, and Logarithmic Functions.*

455. $e^{ix} = \cos x + i \sin x.$

456. $e^{-ix} = \cos x - i \sin x.$

457. $(\cos x + i \sin x)(\cos x - i \sin x) = 1.$

458. $e^{2ix} = \dfrac{1 + i \tan x}{1 - i \tan x}.$

459. $\cos x = \frac{1}{2}(e^{ix} + e^{-ix}).$

460. $\sin x = -\frac{1}{2}i(e^{ix} - e^{-ix}),$ or $i \sin x = \frac{1}{2}(e^{ix} - e^{-ix}).$

461. $\tan x = -i\dfrac{e^{2ix} - 1}{e^{2ix} + 1},$ or $i \tan x = \dfrac{e^{2ix} - 1}{e^{2ix} + 1}.$

462. $\cos^{-1} x = -i \log_e (x + i\sqrt{1 - x^2}).$

463. $\sin^{-1} x = -i \log_e (ix + \sqrt{1 - x^2}).$

464. $\tan^{-1} x = -\frac{1}{2}i \log_e \dfrac{1 + ix}{1 - ix} = \frac{1}{2}i \log_e \dfrac{1 - ix}{1 + ix} = \frac{1}{2}i \log_e \dfrac{i + x}{i - x}.$

$$\left.\begin{array}{ll} 465. & \cos ix = \frac{1}{2}(e^{x} + e^{-x}) = \operatorname{Cosh} x. \\ 466. & \sin ix = \frac{1}{2}i(e^{x} - e^{-x}) = i \operatorname{Sinh} x. \\ 467. & \tan ix = i\dfrac{(e^{x} - e^{-x})}{e^{x} + e^{-x}} = i \operatorname{Tanh} x. \end{array}\right\} \text{See } 714.$$

468. $e^{x+iy} = e^{x}(\cos y + i \sin y).$

469. $a^{x+iy} = a^{x}[\cos (y \log_e a) + i \sin (y \log_e a)].$

From
$$2 \cos u = e^{iu} + e^{-iu},$$
$$2i \sin u = e^{iu} - e^{-iu},$$
are obtained,

470. $2^{m-1} \cos^m u = \cos mu + C_1 \cos (m-2) u + C_2 \cos (m-4)u + C_3 \cos (m-6) u + \ldots$

471. $(-1)^{\frac{m}{2}} 2^{m-1} \sin^m u = \cos mu - C_1 \cos (m-2) u + C_2 \cos (m-4) u - C_3 \cos (m-6) u + \ldots$ when $m$ is even,

472. $(-1)^{\frac{m-1}{2}} 2^{m-1} \sin^m u = \sin mu - C_1 \sin (m-2) u + C_2 \sin (m-4) u - C_3 \sin (m-6) u + \ldots$ when $m$ is odd,

wherein $C_1, C_2, C_3 \ldots$ are the Binomial Coefficients.

473. $(\cos x \pm i \sin x)^n = \cos nx \pm i \sin nx.$

474. $\sqrt[n]{\cos x \pm i \sin x} = \cos\frac{x+2k\pi}{n} \pm i \sin\frac{x+2k\pi}{n}.$

475. $\sin(x+iy) = \frac{1}{2}(e^y+e^{-y}) \sin x + \frac{1}{2}i\,(e^y - e^{-y}) \cos x.$

476. $\cos(x+iy) = \frac{1}{2}(e^y+e^{-y}) \cos x - \frac{1}{2}i\,(e^y - e^{-y}) \sin x.$

477. $\log_e(x \pm iy) = \frac{1}{2}\log_e(x^2+y^2) + i\left(\tan^{-1}\frac{y}{x} \pm 2k\pi\right),$

when $x$ is positive.

478. $\log_e(x \pm iy) = \frac{1}{2}\log_e(x^2+y^2) + i\left(\tan^{-1}\frac{y}{x} \pm (2k+1)\pi\right),$

when $x$ is negative.

479. $\log_e\left(\frac{x+iy}{x-iy}\right) = 2i\,(\tan^{-1}\frac{y}{x} \pm 2k\pi).$

480. $\begin{cases} \log_e(+1) = \pm 2k\pi i, \\ \log_e i \quad\;\; = \pm(2k+\frac{1}{2})\pi i, \\ \log_e(-1) = \pm(2k+1)\pi i, \\ \log_e(-i) = \pm(2k+\frac{3}{2})\pi i. \end{cases}$

481. $\begin{cases} e^{2k\pi i} = 1 = i^0, \\ e^{(2k+\frac{1}{2})\pi i} = i = i^1, \\ e^{(2k+1)\pi i} = -1 = i^2, \\ e^{(2k+\frac{3}{2})\pi i} = -i = i^3. \end{cases}$ 482. $\begin{cases} e^x = e^{x+2k\pi i}, \\ \quad = -ie^{x+(2k+\frac{1}{2})\pi i}, \\ \quad = -e^{x+(2k+1)\pi i}, \\ \quad = ie^{x+(2k+\frac{3}{2})\pi i}. \end{cases}$

Let $z$ be any variable, real or imaginary; and let $r$ be its modulus and $\theta$ its argument. Then,

483. $z = r(\cos\theta + i\sin\theta) = re^{\theta i} = re^{(\theta+2k\pi)i},$

484. $\log_e z = \log_e r + (\theta + 2k\pi)\,i.$

## PLANE TRIANGLES.

### *General Properties.*

Formulas expressing the general properties of triangles usually occur in sets of three, of which only one needs to be printed. The others are obtained from it by a *cyclic change of letters*, that is, by changing $a$ to $b$, $b$ to $c$, and $c$ to $a$, also $A$ to $B$, $B$ to $C$, and $C$ to $A$, always in this fixed order. This process applied to the first equation of **501** gives the second, applied to the second gives the third, and applied to the third gives the first again. And so in all cases.

**501.** $$\begin{cases} a = b \cos C + c \cos B, \\ b = c \cos A + a \cos C, \\ c = a \cos B + b \cos A. \end{cases}$$

All the relations between the six parts of a plane triangle are implicitly contained in, and can, by algebraic transformations, be derived from the three equations, **501**.

**502.** $A + B + C = 180°.$

**503.** $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

**504.** $$\frac{a+b}{a-b} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)} \quad \ldots \; ③.^*$$

**505.** $$\frac{a+b}{c} = \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} \quad \ldots \; ③.$$

**506.** $$\frac{a-b}{c} = \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)} \quad \ldots \; ③.$$

**507.** $a^2 = b^2 + c^2 - 2bc \cos A \quad \ldots \; ③.$

**508.** $a^2 + b^2 + c^2 = 2bc \cos A + 2ca \cos B + 2ab \cos C.$

* The symbol ③ indicates that there is a full set of three equations of which only one is printed, the other two being obtainable from this one by a cyclic change of letters.

If $d$ denote the diagonal of a parallelogram drawn from the point where the sides $a$ and $b$ meet making an angle $C$ with each other,

**509.** $$d^2 = a^2 + b^2 + 2ab\cos C.$$

**510.** Let $$s = \tfrac{1}{2}(a + b + c),$$

whence,
$$s - a = \tfrac{1}{2}(-a + b + c),$$
$$s - b = \tfrac{1}{2}(a - b + c),$$
$$s - c = \tfrac{1}{2}(a + b - c).$$

**511.** $$\sin \tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}} \quad \ldots (3).$$

**512.** $$\cos \tfrac{1}{2}A = \sqrt{\frac{s(s-a)}{bc}} \quad \ldots (3).$$

**513.** $$\tan \tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \quad \ldots (3).$$

Let $r$ denote the radius of the inscribed circle.

**514.** $$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}},$$
$$= s\tan \tfrac{1}{2}A \tan \tfrac{1}{2}B \tan \tfrac{1}{2}C,$$
$$= \frac{a \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C}{\cos \tfrac{1}{2}A} \quad \ldots (3).$$

**515.** $$\tan \tfrac{1}{2}A = \frac{r}{s-a} \quad \ldots (3).$$

Let $T$ denote the area of the triangle.

**516.** $$T = sr = \sqrt{s(s-a)(s-b)(s-c)},$$
$$= \tfrac{1}{4}\sqrt{2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4},$$
$$= \frac{2abc}{a+b+c}\cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C,$$
$$= \tfrac{1}{4}(a+b+c)^2 \tan \tfrac{1}{2}A \tan \tfrac{1}{2}B \tan \tfrac{1}{2}C,$$
$$= \frac{c^2 \sin A \sin B}{2\sin(A+B)} = \tfrac{1}{2}ab\sin C \quad \ldots (3).$$

**517.** $$\sin A = \frac{2T}{bc} = \frac{2sr}{bc} \quad \ldots (3).$$

Let $p_a$, $p_b$, $p_c$ denote perpendiculars from the vertices upon the sides $a$, $b$, and $c$ respectively. Then

518. $p_a = b \sin C = c \sin B = \dfrac{bc \sin A}{a} \ldots ③,$

$$= \frac{\sin B \sin C}{\sin A} a = \frac{2T}{a} \ldots ③.$$

Let $r_a$, $r_b$, and $r_c$ denote the radii of the three escribed circles touching externally the sides $a$, $b$, and $c$ respectively.

519. $r_a = s \tan \frac{1}{2}A = \dfrac{T}{s-a} = \dfrac{sr}{s-a} = \dfrac{a \cos \frac{1}{2}B \cos \frac{1}{2}C}{\cos \frac{1}{2} A} \ldots ③.$

520. $T = \sqrt{r\,r_a r_b r_c}.$

521. $r_a r_b r_c = abc \cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C.$

522. $r = \sqrt{r_a r_b} + \sqrt{r_b r_c} + \sqrt{r_c r_a}.$

523. $\tan \frac{1}{2}A = \sqrt{\dfrac{r\,r_a}{r_b r_c}} \ldots ③.$

524. $\dfrac{1}{r_a} + \dfrac{1}{r_b} + \dfrac{1}{r_c} = \dfrac{1}{r} = \dfrac{1}{p_a} + \dfrac{1}{p_b} + \dfrac{1}{p_c}.$

Let $R$ denote the radius of the circumscribed circle.

525. $R = \frac{1}{2}a \csc A$

$$= \tfrac{1}{4}\sqrt{(b+c)^2 \sec^2 \tfrac{1}{2}A + (b-c)^2 \csc^2 \tfrac{1}{2}A} \ldots ③,$$

$$= \tfrac{1}{4}s \sec \tfrac{1}{2}A \sec \tfrac{1}{2}B \sec \tfrac{1}{2}C,$$

$$= \tfrac{1}{4}\,(r_a + r_b + r_c - r) = \frac{abc}{4T}.$$

526. $2Rr = \dfrac{abc}{a+b+c}.$

527. $R + r = \frac{1}{2}\,(a \operatorname{ctn} A + b \operatorname{ctn} B + c \operatorname{ctn} C),$

= sum of the perpendiculars to the sides from the centre of the circumscribing circle.

528. $\sqrt{R^2 - 2Rr}$ = distance between the centres of the inscribed and circumscribed circles.

If O be the centre of the inscribed circle its distance from the vertex A of the triangle is

529. $$\text{OA} = \frac{2bc}{a+b+c}\cos\tfrac{1}{2}A \;.\;.\;.\; ③.$$

530. $$\cos A + \cos B + \cos C = 1 + \frac{2a\sin B\sin C}{a+b+c} \;.\;.\;.\; ③.$$

531. $$a\cos A + b\cos B + c\cos C = 4R\sin A\sin B\sin C, \quad = 2a\sin B\sin C \;.\;.\;.\; ③.$$

If $A + B + C = 180°$, then follow **532–540**,

532. $\sin A + \sin B + \sin C = 4\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C$,

533. $\sin A + \sin B - \sin C = 4\sin\frac{1}{2}A\sin\frac{1}{2}B\cos\frac{1}{2}C$,

534. $\cos A + \cos B + \cos C = 4\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C + 1$,

535. $\cos A + \cos B - \cos C = 4\cos\frac{1}{2}A\cos\frac{1}{2}B\sin\frac{1}{2}C - 1$,

536. $\tan A + \tan B + \tan C = \tan A\tan B\tan C$,

537. $\operatorname{ctn}\frac{1}{2}A + \operatorname{ctn}\frac{1}{2}B + \operatorname{ctn}\frac{1}{2}C = \operatorname{ctn}\frac{1}{2}A\operatorname{ctn}\frac{1}{2}B\operatorname{ctn}\frac{1}{2}C$,

538. $\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C$,

539. $\cos 2A + \cos 2B + \cos 2C = -4\cos A\cos B\cos C - 1$,

540. $\operatorname{ctn} A\operatorname{ctn} B + \operatorname{ctn} B\operatorname{ctn} C + \operatorname{ctn} C\operatorname{ctn} A = 1$.

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If $A + B + C = 90°$,

541. $\tan A\tan B + \tan B\tan C + \tan C\tan A = 1$.

*The Quadrilateral Inscribed in a Circle.*

542. Angles, $A + C = 180°$, $B + D = 180°$. Sides, AB $= a$, BC $= b$, CD $= c$, DA $= d$.

543. $$\cos B = \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}.$$

544. Diagonal, $$\overline{\text{AC}}^2 = \frac{(ac + bd)(ad + bc)}{ab + cd}.$$

Put $s = \frac{1}{2}(a + b + c + d)$,

545. $$Q = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$
= area of the quadrilateral.

546. $\sin A = \sin C = \dfrac{2Q}{bc + da}$.

547. $\sin B = \sin D = \dfrac{2Q}{ab + cd}$.

Radius of the circumscribed circle,

548. $R = \dfrac{1}{4Q}\sqrt{(ab + cd)\,(ac + bd)\,(ad + bc)}$.

*Solution of Right Triangles.*

549. CASE I. Given an angle and the hypotenuse, $A$ and $h$, to find $B$, $a$, and $b$.

$$B = 90° - A.$$

$$a = h \sin A. \qquad \log a = \log h + \log \sin A.$$
$$b = h \cos A. \qquad \log b = \log h + \log \cos A.$$

TEST. The computed values of $a$ and $b$ should satisfy

$$2 \log a = \log (h + b) + \log (h - b),$$
$$2 \log b = \log (h + a) + \log (h - a).$$

550. CASE II. Given an angle and the leg opposite, $A$ and $a$, to find $B$, $h$, and $b$.

$$B = 90° - A.$$

$$h = a \csc A. \qquad \log h = \log a + \log \csc A.$$
$$b = a \operatorname{ctn} A. \qquad \log b = \log a + \log \operatorname{ctn} A.$$

TEST. The computed values of $h$ and $b$ should satisfy

$$2 \log a = \log (h + b) + \log (h - b),$$
$$2 \log b = \log (h + a) + \log (h - a).$$

551. CASE III. Given an angle and the leg adjacent, $A$ and $b$, to find $B$, $h$, and $a$.

$$B = 90° - A.$$

$$h = b \sec A. \qquad \log h = \log b + \log \sec A.$$
$$a = b \tan A. \qquad \log a = \log b + \log \tan A.$$

TEST. The computed values of $h$ and $a$ should satisfy

$$2 \log a = \log (h + b) + \log (h - b),$$
$$2 \log b = \log (h + a) + \log (h - a).$$

552. Case IV. Given the hypotenuse and a leg, $h$ and $a$, to find $A$, $B$, and $b$.

$$\sin A = \cos B = \frac{a}{h}, \qquad \log \sin A = \log \cos B$$
$$= \log a + \text{co-log } h - 10.$$

whence $A = \sin^{-1}\frac{a}{h}$,

$$B = \cos^{-1}\frac{a}{h}.$$

$$b = \sqrt{(h+a)(h-a)}. \quad \log b = \tfrac{1}{2}[\log(h+a) + \log(h-a)].$$

Test. The computed values of $A$ and $b$ should satisfy

$$\log b + \log \tan A = \log a.$$

553. Case V. Given the two legs, $a$ and $b$, to find $A$, $B$, and $h$.

$$\tan A = \frac{a}{b}. \qquad \log \tan A = \log \text{ctn } B$$
$$= \log a + \text{co-log } b - 10.$$

$$\text{ctn } B = \frac{a}{b}.$$

$$A = \tan^{-1}\frac{a}{b}.$$

$$B = \text{ctn}^{-1}\frac{a}{b}.$$

$$h = a \csc A, \qquad \log h = \log a + \log \csc A.$$
$$= \sqrt{a^2 + b^2}.$$

Test. The computed value of $h$ should satisfy

$$2 \log a = \log(h+b) + \log(h-b),$$
$$2 \log b = \log(h+a) + \log(h-a).$$

*Special Formulas for Plane Right Triangles.*

554. $$h - b = 2h \sin^2 \tfrac{1}{2}A,$$

gives $h - b$ with great accuracy when $A$ is small, or $A$ with great accuracy when $h$ and $b$ are nearly equal.

555. $$\tan \tfrac{1}{2}A = \sqrt{\frac{h-b}{h+b}} = \frac{h-b}{a}.$$

556. $\tan(45^\circ \pm A) = \dfrac{b \pm a}{b \mp a}$

557. $\sin(B - A) = \dfrac{(b+a)(b-a)}{h^2}$, $\cos(B - A) = \dfrac{2ab}{h^2}$.

558. $\tan(B - A) = \dfrac{(b+a)(b-a)}{2ab}$,

a formula which gives $B - A$ with great accuracy when $a$ and $b$ are given nearly equal, or $b - a$ with great accuracy when $A$ and $B$ are given nearly equal.

For the use of $S$ and $T$, and a table of the values of these functions up to 2°, see page **83**.

*Solution of Plane Oblique Triangles.*

An oblique triangle can be solved in two ways, (*i*) by converting it into the sum or the difference of two right triangles, formed by drawing a perpendicular from any vertex, and solving these right triangles by methods above given (549–553), or (*ii*) by substituting the given parts in general formulas, and working out the required parts. Either method serves well to test the accuracy of the results obtained by the other. The outlines of solutions by both methods are given in the following formulas.

To preserve a uniformity of notation let the perpendiculars from the vertices A, B, and C upon the sides $a$, $b$, and $c$ of the triangle be marked AP, BQ, CR respectively; so that always

559. $a = \text{BP} + \text{PC}$, $b = \text{CQ} + \text{QA}$, $c = \text{AR} + \text{RB}$.

These equations hold as well when the perpendicular falls without as when it falls within the triangle, if regard be had to the principle stated in 335.

CASE I. Given two angles and a side, $A$, $B$, and $c$, to find $a$, $b$, and $C$.

$$C = 180^\circ - (A + B).$$

**560.** First Method. By drawing a perpendicular from either end of the given side.

| Perpendicular AP. | Perpendicular BQ. |
|---|---|
| $\text{AP} = c \sin B.$ | $\text{BQ} = c \sin A.$ |
| $\text{BP} = c \cos B.$ | $\text{AQ} = c \cos A.$ |
| $\text{PC} = \text{AP} \operatorname{ctn} C.$ | $\text{QC} = \text{BQ} \operatorname{ctn} C.$ |
| $a = \text{BP} + \text{PC}.$ | $b = \text{AQ} + \text{QC}.$ |
| $b = \text{AP} \csc C.$ | $a = \text{BQ} \csc C.$ |

Either process affords a test of the results obtained by the other.

**561.** Second Method. By the general formulas.

$$a = 2R \sin A.$$
$$b = 2R \sin B.$$
$$c = 2R \sin C.$$

Compute $2R$ (or $\log 2R$) by aid of the last of these equations, then $a$ and $b$ by aid of the other two.

**562.** Otherwise by the formulas

$$\left.\begin{aligned} a + b &= c\frac{\cos \frac{1}{2}(A - B)}{\cos \frac{1}{2}(A + B)} = S. \\ a - b &= c\frac{\sin \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B)} = D. \\ a &= \tfrac{1}{2}(S + D). \\ b &= \tfrac{1}{2}(S - D). \end{aligned}\right\}$$

which, being in different form, afford a good test of the results obtained by the formulas first used.

Case II. Given two sides and the included angle, $a$, $b$, and $C$, to find $A$, $B$, and $c$.

**563.** First Method. By drawing a perpendicular to the longer given side from the end of the shorter. Let $b > a$.

Perpendicular BQ.

$$\text{BQ} = a \sin C. \qquad \tan A = \frac{\text{BQ}}{\text{QA}}, \text{ or } A = \tan^{-1}\frac{\text{BQ}}{\text{QA}}.$$

$$\text{CQ} = a \cos C. \qquad B = 180° - (C + A).$$

$$\text{QA} = b - \text{CQ}. \qquad c = \text{BQ} \csc A = \text{QA} \sec A.$$

564. SECOND METHOD. By the general formulas.

$$\tfrac{1}{2}(A+B) = 90^\circ - \tfrac{1}{2}C.$$

$$\tan\tfrac{1}{2}(A-B) = \frac{a-b}{a+b}\tan\tfrac{1}{2}(A+B).$$

$$A = \tfrac{1}{2}(A+B) + \tfrac{1}{2}(A-B).$$

$$B = \tfrac{1}{2}(A+B) - \tfrac{1}{2}(A-B).$$

$$c = \frac{a\sin C}{\sin A} = \frac{b\sin C}{\sin B} = 2R\sin C.$$

Or, as a test, $c$ may be computed by either of the equations

$$c = (a+b)\frac{\cos\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A-B)} = (a-b)\frac{\sin\frac{1}{2}(A+B)}{\sin\frac{1}{2}(A-B)}.$$

CASE III. Given two sides and the angle opposite one of them $a$, $b$, and $A$, to find $B$, $C$, and $c$.

565. FIRST METHOD. By drawing a perpendicular fron the point where the given sides meet.

Perpendicular CR.

$$\text{CR} = b\sin A.$$

$$\text{AR} = b\cos A.$$

$$\text{RB} = \sqrt{(a+\text{CR})(a-\text{CR})}.$$

$$c = \text{AR} \pm \text{RB}, \text{ a double solution.}$$

$$\left.\begin{aligned} c_1 &= \text{AR} + \text{RB} \\ c_2 &= \text{AR} - \text{RB} \end{aligned}\right\}.$$

$$\text{Angle ACR} = 90^\circ - A.$$

$$\text{Angle RCB} = \sin^{-1}\frac{\text{RB}}{a} = \cos^{-1}\frac{\text{CR}}{a}.$$

$$\left.\begin{aligned} C_1 &= \text{ACR} + \text{RCB} \\ C_2 &= \text{ACR} - \text{RCB} \end{aligned}\right\}.$$

$$\left.\begin{aligned} B_1 &= 180^\circ - (A + C_1) = A + C_2 \\ B_2 &= 180^\circ - (A + C_2) = A + C_1 \end{aligned}\right\}.$$

If $\text{CR} = b\sin A < a$, there is no real solution.

**566.** Second Method. By the general formulas.

$$\sin B = \frac{b \sin A}{a}, \text{ giving two values of } B,\ B_1 < 90° < B_2.$$

$$C_1 = B_2 - A, \qquad C_2 = B_1 - A.$$

$$c_1 = \frac{a \sin C_1}{\sin A}, \qquad c_2 = \frac{a \sin C_2}{\sin A}.$$

If $\sin B$ comes out greater than 1, there is no real solution.

If $C_2$ comes out negative the second solution is inadmissible.

Case IV. Given the three sides $a$, $b$, and $c$, to find the angles $A$, $B$, and $C$.

**567.** First Method. By drawing a perpendicular to the longest side, or to the side that is not less than either of the other two.

Let $c > b > a$. Or, let $c$ be not less than either $b$ or $a$.

Perpendicular CR.

$$\text{AR} + \text{RB} = c.$$

$$\text{AR} - \text{RB} = \frac{(b + a)(b - a)}{c}.$$

$$\cos A = \frac{\text{AR}}{b}, \text{ or } A = \cos^{-1}\frac{\text{AR}}{b}.$$

$$\cos B = \frac{\text{RB}}{a}, \text{ or } B = \cos^{-1}\frac{\text{RB}}{a}.$$

$$C = 180° - (A + B).$$

Test, by drawing another perpendicular, which *may* fall outside the triangle.

Perpendicular AP (which is supposed to fall outside the triangle),

$$\text{BP} - \text{CP} = a.$$

$$\text{BP} + \text{CP} = \frac{(c + b)(c - b)}{a}.$$

$$\cos B = \frac{\text{BP}}{c}, \text{ or } B = \cos^{-1}\frac{\text{BP}}{c}.$$

$$\cos C = \frac{-\text{CP}}{b}, \text{ or } C = \cos^{-1}\left(\frac{-\text{CP}}{b}\right) > 90°.$$

$$A = 180° - (B + C).$$

**568.** Second Method. By the general formulas.

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

$$\tan \tfrac{1}{2}A = \frac{r}{s-a}, \quad \tan \tfrac{1}{2}B = \frac{r}{s-b}, \quad \tan \tfrac{1}{2}C = \frac{r}{s-c}.$$

Test. $A + B + C = 180°$.

The half-angles may also be computed by

$$\sin \tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}} \quad . \; . \; . (3).$$

or by

$$\cos \tfrac{1}{2}A = \sqrt{\frac{s(s-a)}{bc}} \quad . \; . \; . (3).$$

Special Formulas for the case of two nearly equal sides or angles.

**569.** $$a - b = \frac{2a \cos \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B)}{\sin A} \quad . \; . \; . (3).$$

**570.** $$\sin \tfrac{1}{2}(A-B) = \frac{(a-b)\sin A}{2a \cos \frac{1}{2}(A+B)} \quad . \; . \; . (3).$$

For the use of $S$ and $T$ and a table of the values of these functions up to 2°, see page 83.

## SPHERICAL TRIANGLES.

### *General Properties.*

Let $a$, $b$, $c$, denote the sides and $A$, $B$, $C$, the angles of a spherical triangle; and let $a'$, $b'$, $c'$, denote the sides and $A'$, $B'$, $C'$, the angles of its polar triangle. Then

**601.**
$$a + A' = 180^\circ. \qquad A + a' = 180^\circ.$$
$$b + B' = 180^\circ. \qquad B + b' = 180^\circ.$$
$$c + C' = 180^\circ. \qquad C + c' = 180^\circ.$$

The perpendicular great-circle arcs AP, BQ, CR, drawn in the spherical triangle coincide with the perpendicular great-circle arcs A'P', B'Q', C'R', similarly drawn in its polar triangle.

The fundamental equations, from which are or can be derived all other general equations relating to spherical triangles, are these three,

**602.**
$$\begin{cases} \cos a = \cos b \cos c + \sin b \sin c \cos A, \\ \cos b = \cos c \cos a + \sin c \sin a \cos B, \\ \cos c = \cos a \cos b + \sin a \sin b \cos C. \end{cases}$$

**603.**
$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} = M = \textit{The Modulus.}$$

**604.**
$$\begin{cases} \cos A = -\cos B \cos C + \sin B \sin C \cos a, \\ \cos B = -\cos C \cos A + \sin C \sin A \cos b, \\ \cos C = -\cos A \cos B + \sin A \sin B \cos c. \end{cases}$$

**605.**
$$\begin{cases} \operatorname{ctn} a \sin b = \cos b \cos C + \sin C \operatorname{ctn} A, \\ \operatorname{ctn} b \sin c = \cos c \cos A + \sin A \operatorname{ctn} B \\ \operatorname{ctn} c \sin a = \cos a \cos B + \sin B \operatorname{ctn} C, \\ \\ \operatorname{ctn} a \sin c = \cos c \cos B + \sin B \operatorname{ctn} A, \\ \operatorname{ctn} b \sin a = \cos a \cos C + \sin C \operatorname{ctn} B, \\ \operatorname{ctn} c \sin b = \cos b \cos A + \sin A \operatorname{ctn} C. \end{cases}$$

606. $$\begin{cases} \sin a \cos B = \sin c \cos b - \cos c \sin b \cos A, \\ \sin b \cos C = \sin a \cos c - \cos a \sin c \cos B, \\ \sin c \cos A = \sin b \cos a - \cos b \sin a \cos C, \\ \\ \sin a \cos C = \sin b \cos c - \cos b \sin c \cos A, \\ \sin b \cos A = \sin c \cos a - \cos c \sin a \cos B, \\ \sin c \cos B = \sin a \cos b - \cos a \sin b \cos C. \end{cases}$$

607. $$\begin{cases} \sin A \cos b = \sin C \cos B + \cos C \sin B \cos a, \\ \sin B \cos c = \sin A \cos C + \cos A \sin C \cos b, \\ \sin C \cos a = \sin B \cos A + \cos B \sin A \cos c, \\ \\ \sin A \cos c = \sin B \cos C + \cos B \sin C \cos a, \\ \sin B \cos a = \sin C \cos A + \cos C \sin A \cos b, \\ \sin C \cos b = \sin A \cos B + \cos A \sin B \cos c. \end{cases}$$

608. $\cos a = \cos (b + c) + 2 \sin b \sin c \cos^2 \tfrac{1}{2}A \quad . . . ③,$
$\quad = \cos (b - c) - 2 \sin b \sin c \sin^2 \tfrac{1}{2}A \quad . . . ③.$

609. $\cos A = - \cos (B + C) - 2 \sin B \sin C \sin^2 \tfrac{1}{2}a \quad . . . ③,$
$\quad = - \cos (B - C) + 2 \sin B \sin C \cos^2 \tfrac{1}{2}a \quad . . . ③.$

610. $s = \tfrac{1}{2} (a + b + c).$

611. $\sin \tfrac{1}{2}A = \sqrt{\dfrac{\sin (s - b) \sin (s - c)}{\sin b \sin c}} \quad . . . ③.$

612. $\cos \tfrac{1}{2}A = \sqrt{\dfrac{\sin s \sin (s - a)}{\sin b \sin c}} \quad . . . ③.$

613. $\tan \tfrac{1}{2}A = \sqrt{\dfrac{\sin (s - b) \sin (s - c)}{\sin s \sin (s - a)}} \quad . . . ③.$

614. $S = \tfrac{1}{2} (A + B + C).$

Spherical Excess of a Spherical Triangle $= 2E$.

615. $2E = A + B + C - 180° = 2S - 180°.$

616. $\cos \tfrac{1}{2}a = \sqrt{\dfrac{\sin (B - E) \sin (C - E)}{\sin B \sin C}} \quad . . . ③,$

$\quad = \sqrt{\dfrac{\cos (S - B) \cos (S - C)}{\sin B \sin C}} \quad . . . ③.$

**617.** $\sin\ \frac{1}{2}a = \sqrt{\dfrac{\sin E \sin (A-E)}{\sin B \sin C}} \ldots (3),$

$= \sqrt{\dfrac{-\cos S \cos (S-A)}{\sin B \sin C}} \ldots (3).$

**618.** $\operatorname{ctn}\ \frac{1}{2}a = \sqrt{\dfrac{\sin (B-E) \sin (C-E)}{\sin E \sin (A-E)}} \ldots (3),$

$= \sqrt{\dfrac{\cos (S-B) \cos (S-C)}{-\cos S \cos (S-A)}} \ldots (3).$

**619.** $\tan \frac{1}{2}E = \sqrt{\tan \frac{1}{2}s \tan \frac{1}{2}(s-a) \tan \frac{1}{2}(s-b) \tan \frac{1}{2}(s-c)}.$

Let $r$ denote the polar radius of the circle inscribed in the spherical triangle.

**620.** $\tan r = \sqrt{\dfrac{\sin (s-a) \sin (s-b) \sin (s-c)}{\sin s}}.$

Let $R$ denote the polar radius of the circle circumscribed about the spherical triangle.

**621.** $\operatorname{ctn} R = \sqrt{\dfrac{\sin (A-E) \sin (B-E) \sin (C-E)}{\sin E}}$

**622.** $\tan \frac{1}{2}A = \dfrac{\tan r}{\sin (s-a)} \ldots (3).$

**623.** $\operatorname{ctn} \frac{1}{2}a = \dfrac{\operatorname{ctn} R}{\sin (A-E)} \ldots (3).$

**624.** $\operatorname{ctn} R = \sqrt{\dfrac{\cos (S-A) \cos (S-B) \cos (S-C)}{-\cos S}}.$

**625.** $\operatorname{ctn} \frac{1}{2}a = \dfrac{\operatorname{ctn} R}{\cos (S-A)} \ldots (3).$

**626.** $2 \tan r \sin s = \sin A \sin b \sin c \ldots (3).$

**627.** $2 \operatorname{ctn} R \sin E = \sin a \sin B \sin C \ldots (3).$

**628.** $M = \text{the modulus} = \dfrac{\tan r \sin s}{\operatorname{ctn} R \sin E}.$

629. $$\tan E = \frac{\tan \frac{1}{2}a \tan \frac{1}{2}b \sin C}{1 + \tan \frac{1}{2}a \tan \frac{1}{2}b \cos C} \ldots (3),$$

$$= \frac{2 \tan r \sin s}{1 + \cos a + \cos b + \cos c}.$$

Let $p_a$, $p_b$, $p_c$, denote the perpendicular great-circle arcs upon the sides $a$, $b$, $c$, respectively from the opposite vertices.

630. $$\sin p_a = \sin b \sin C = \sin c \sin B \ (3),$$

$$= \frac{\sin b \sin c \sin A}{\sin a} = \frac{\sin B \sin C \sin a}{\sin A} \ (3),$$

$$= \frac{2 \tan r \sin s}{\sin a} = \frac{2 \operatorname{ctn} R \sin E}{\sin A} \ldots (3).$$

Let $r_a$, $r_b$, $r_c$, denote the radii of the circles inscribed in the three triangles formed by prolonging the sides of the original triangle till they meet in points diametrically opposite its vertices.

Let $R_a$, $R_b$, $R_c$ denote the radii of the circles circumscribed about these same triangles.

631. $$\tan r_a = \sin s \tan \tfrac{1}{2}A = \frac{\sin s \tan r}{\sin (s - a)} \ldots (3).$$

632. $$\operatorname{ctn} R_a = \sin E \operatorname{ctn} \tfrac{1}{2}a = \frac{\sin E \operatorname{ctn} R}{\sin (A - E)} \ldots (3).$$

*Cagnoli's Equation.*

633.
$$\sin a \sin b + \cos a \cos b \cos C = \sin A \sin B - \cos A \cos B \cos c.$$

Either member of this equation has the same value in a given spherical triangle and in its polar triangle.

*Gauss's Equations.*

634.
$$\begin{cases} \text{(i)} & \dfrac{\sin\frac{1}{2}(A+B)}{\cos\frac{1}{2}C}=\dfrac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}c}, \\ \text{(ii)} & \dfrac{\sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}C}=\dfrac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}c}, \\ \text{(iii)} & \dfrac{\cos\frac{1}{2}(A+B)}{\sin\frac{1}{2}C}=\dfrac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}c}, \\ \text{(iv)} & \dfrac{\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}C}=\dfrac{\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}c}. \end{cases}$$

*Napier's Analogies.*

635.
$$\begin{cases} \text{(i)} & \dfrac{\sin\frac{1}{2}(A+B)}{\sin\frac{1}{2}(A-B)}=\dfrac{\tan\frac{1}{2}c}{\tan\frac{1}{2}(a-b)}, \\ \text{(ii)} & \dfrac{\cos\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A-B)}=\dfrac{\tan\frac{1}{2}c}{\tan\frac{1}{2}(a+b)}, \\ \text{(iii)} & \dfrac{\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}(a-b)}=\dfrac{\operatorname{ctn}\frac{1}{2}C}{\tan\frac{1}{2}(A-B)}, \\ \text{(iv)} & \dfrac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}(a-b)}=\dfrac{\operatorname{ctn}\frac{1}{2}C}{\tan\frac{1}{2}(A+B)}. \end{cases}$$

*Ten Equations for Spherical Right Triangles.*

Derived from the general equations by making $C = 90°$ and writing $h$ for $c$. Usually cited as Napier's Rules.

636.
$$\begin{cases} \sin a=\sin h\sin A, & \cos A=\cos a\sin B, \\ \sin a=\tan b\operatorname{ctn} B, & \cos A=\operatorname{ctn} h\tan b, \\ \sin b=\sin h\sin B, & \cos B=\cos b\sin A, \\ \sin b=\tan a\operatorname{ctn} A, & \cos B=\operatorname{ctn} h\tan a, \\ \multicolumn{2}{c}{\cos h=\cos a\cos b,} \\ \multicolumn{2}{c}{\cos h=\operatorname{ctn} A\operatorname{ctn} B.} \end{cases}$$

*Ten Equations for Spherical Quadrantal Triangles.*

Derived from the general equations by making $c = 90°$.

637.
$$\begin{cases} \sin A=\sin C\sin a, & \cos a=\cos A\sin b, \\ \sin A=\tan B\operatorname{ctn} b, & \cos a=-\operatorname{ctn} C\tan B, \\ \sin B=\sin C\sin b, & \cos b=\cos B\sin a, \\ \sin B=\tan A\operatorname{ctn} a, & \cos b=-\operatorname{ctn} C\tan A, \\ \multicolumn{2}{c}{\cos C=-\cos A\cos B,} \\ \multicolumn{2}{c}{\cos C=-\operatorname{ctn} a\operatorname{ctn} b.} \end{cases}$$

*The Solution of Spherical Right Triangles.*

**638.** CASE I. Given $A$ and $h$, to find $a$, $b$, and $B$.

For solution,

$\sin a = \sin A \sin h,$

$\tan b = \cos A \tan h,$

$\operatorname{ctn} B = \tan A \cos h.$

For a test,

$\sin a = \tan b \operatorname{ctn} B.$

Two values of each computed part,

$a_2 = 180° - a_1,$

$b_2 = 180° + b_1,$

$B_2 = 180° + B_1.$

For deciding which of the computed values belong together,

$$\tan A = \frac{\tan a}{\sin b},$$

$$\cos A = \cos a \sin B.$$

**639.** CASE II. Given $A$ and $a$, to find $h$, $b$, and $B$.

For solution,

$\sin h = \csc A \sin a,$

$\sin b = \operatorname{ctn} A \tan a,$

$\sin B = \cos A \sec a.$

For a test,

$\sin b = \sin B \sin h.$

No solution if

$\pm (a - 90°) < \pm (A - 90°);$

that is, if $\sin a > \sin A$.

Two values of each computed part,

$h_2 = 180° - h_1,$

$b_2 = 180° - b_1$, or $540° - b_1,$

$B_2 = 180° - B_1$, or $540° - B_1.$

For deciding which of the computed values belong together,

$$\cos a = \frac{\cos h}{\cos b},$$

$$\tan a = \tan h \cos B.$$

**640.** CASE III. Given $A$ and $b$, to find $a$, $h$, and $B$.

For solution,

$\tan a = \tan A \sin b,$

$\operatorname{ctn} h = \cos A \operatorname{ctn} b,$

$\cos B = \sin A \cos b.$

For a test,

$\tan a \operatorname{ctn} h = \cos B.$

Two values of each computed part,

$a_2 = 180° + a_1,$

$h_2 = 180° + h_1,$

$B_2 = 360° - B_1.$

For deciding which of the computed values belong together,

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \sin h.$$

**641.** Case IV. Given $h$ and $a$, to find $A$, $b$, and $B$.

For solution,

$\sin A = \csc h \sin a,$
$\cos b = \cos h \sec a,$
$\cos B = \operatorname{ctn} h \tan a.$

For a test,

$\sin A \cos b = \cos B.$

Two values of each computed part,

$A_2 = 180° - A_1,$
$b_2 = 360° - b_1,$
$B_2 = 360° - B_1.$

For deciding which of the computed values belong together,

$\tan a = \sin b \tan A,$

$$\cos a = \frac{\cos A}{\sin B}.$$

No solution if $a < 90°$ and either $h < a$ or $h > 180° - a$;
or if $a > 90°$ and either $h > a$ or $h < 180° - a$.

**642.** Case V. Given $a$ and $b$, to find $h$, $A$, and $B$.

For solution,

$\cos h = \cos a \cos b,$
$\operatorname{ctn} A = \operatorname{ctn} a \sin b,$
$\operatorname{ctn} B = \sin a \operatorname{ctn} b.$

For a test,

$\operatorname{ctn} A \operatorname{ctn} B = \cos h.$

Two values of each computed part,

$h_2 = 360° - h_1,$
$A_2 = 180° + A_1,$
$B_2 = 180° + B_1.$

For deciding which of the computed values belong together,

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \sin h.$$

**643.** Case VI. Given $A$ and $B$, to find $h$, $a$, and $b$.

For solution,

$\cos h = \operatorname{ctn} A \operatorname{ctn} B,$
$\cos a = \cos A \csc B,$
$\cos b = \csc A \cos B.$

For a test,

$\cos a \cos b = \cos h.$

Two values of each computed part,

$h_2 = 360° - h_1,$
$a_2 = 360° - a_1,$
$b_2 = 360° - b_1.$

For deciding which of the computed values belong together,

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \sin h.$$

*Solution of Spherical Oblique Triangles.*

**644.** CASE I. Given two sides and the included angle, $a$, $b$, and $C$, to find $A$, $B$, and $c$.

Two triangles always possible.

$$A_2 = 180° + A_1,$$
$$B_2 = 180° + B_1,$$
$$c_2 = 360° - c_1.$$

Which of the computed values belong together as parts of one triangle and which as parts of the other is decided by means of the equations

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} = M.$$

The computed values of $A$, $B$, and $c$, which, with the given values of $a$, $b$, and $C$, make each of these ratios positive, belong to one triangle, and those which make each ratio negative belong to the other triangle. In other words, one triangle has a positive, the other a negative modulus.

**645.** FIRST METHOD. By **605** and **602**,

$$\operatorname{ctn} A = \frac{\cos a \sin b - \sin a \cos b \cos C}{\sin a \sin C},$$

$$\operatorname{ctn} B = \frac{\sin a \cos b - \cos a \sin b \cos C}{\sin b \sin C},$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

**646.** SECOND METHOD. Based on Gauss's Equations [**634**], which, by taking the logarithm of each term, may be written thus:

(i) $\log \sin \frac{1}{2}(A + B) + \log \cos \frac{1}{2}c =$
$\log \cos \frac{1}{2}(a - b) + \log \cos \frac{1}{2}C,$

(ii) $\log \sin \frac{1}{2}(A - B) + \log \sin \frac{1}{2}c =$
$\log \sin \frac{1}{2}(a - b) + \log \cos \frac{1}{2}C,$

(iii) $\log \cos \frac{1}{2}(A + B) + \log \cos \frac{1}{2}c =$
$\log \cos \frac{1}{2}(a + b) + \log \sin \frac{1}{2}C,$

(iv) $\log \cos \frac{1}{2}(A - B) + \log \sin \frac{1}{2}c =$
$\log \sin \frac{1}{2}(a + b) + \log \sin \frac{1}{2}C.$

**647.** Third Method. Based on Napier's Analogies [635], which, by taking the logarithm of each term, may be written thus:

(iv) $\log\tan\frac{1}{2}(A+B) =$
$$\log\sec\tfrac{1}{2}(a+b)+\log\cos\tfrac{1}{2}(a-b)+\log\operatorname{ctn}\tfrac{1}{2}C,$$

(iii) $\log\tan\frac{1}{2}(A-B) =$
$$\log\csc\tfrac{1}{2}(a+b)+\log\sin\tfrac{1}{2}(a-b)+\log\operatorname{ctn}\tfrac{1}{2}C,$$

(ii) $\log\tan\frac{1}{2}c =$
$$\log\cos\tfrac{1}{2}(A+B)+\log\sec\tfrac{1}{2}(A-B)+\log\tan\tfrac{1}{2}a+b),$$

(i) $\log\tan\frac{1}{2}c =$
$$\log\sin\tfrac{1}{2}(A+B)+\log\csc\tfrac{1}{2}(A-B)+\log\tan\tfrac{1}{2}(a-b).$$

**648.** Fourth Method. By drawing from B or from A a great-circle arc perpendicular to the opposite side, the required triangle is converted into the sum or the difference of two spherical right triangles.

| Perpendicular BQ. | Perpendicular AP. |
|---|---|
| $\tan \text{CQ} = \tan a\cos C.$ | $\tan \text{CP} = \tan b\cos C.$ |
| $\left.\begin{array}{l}\text{QA} = b - \text{CQ},\\ \text{or } \text{QA} = 360^\circ + b - \text{CQ}.\end{array}\right\}$ | $\left.\begin{array}{l}\text{PB} = a - \text{CP},\\ \text{or } \text{PB} = 360^\circ + a - \text{CP}.\end{array}\right\}$ |
| $\operatorname{ctn} A = \operatorname{ctn} C\csc \text{CQ}\sin \text{QA}.$ | $\operatorname{ctn} B = \operatorname{ctn} C\csc \text{CP}\sin \text{PB}.$ |
| $\cos c = \cos a\sec \text{CQ}\cos \text{QA}.$ | $\cos c = \cos b\sec \text{CP}\cos \text{PB}.$ |

This method is equivalent to that of some writers who introduce an auxiliary angle, $\varphi$ or $\chi$, defined by the equation $\tan\varphi = \tan a\cos C$ or $\tan\chi = \tan b\cos C$.

**649.** Case II. Given two angles and the included side $A$, $B$, and $c$ to find $a$, $b$, and $C$. Two triangles always possible.

$$a_2 = 180^\circ + a_1,$$
$$b_2 = 180^\circ + b_1,$$
$$C_2 = 360^\circ - C_1.$$

Which values belong together is decided by means of the equations

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} = M.$$

The computed values of $a$, $b$, and $C$, which, with the given values of $A$, $B$, and $c$, make each of these ratios positive belong to one triangle, and those which make each ratio negative belong to the other triangle. One triangle has a positive, the other a negative modulus.

FIRST METHOD. By 605 and 604,

$$\operatorname{ctn} a = \frac{\cos A \sin B + \sin A \cos B \cos c}{\sin A \sin c},$$

$$\operatorname{ctn} b = \frac{\sin A \cos B + \cos A \sin B \cos c}{\sin B \sin c},$$

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c.$$

650. SECOND METHOD. — Based on Gauss's Equations (634), which, by taking the logarithm of each term, may be written thus:

(i) $\log \cos \frac{1}{2}(a-b) + \log \cos \frac{1}{2}C = \log \sin \frac{1}{2}(A+B) + \log \cos \frac{1}{2}c,$

(ii) $\log \sin \frac{1}{2}(a-b) + \log \cos \frac{1}{2}C = \log \sin \frac{1}{2}(A-B) + \log \sin \frac{1}{2}c,$

(iii) $\log \cos \frac{1}{2}(a+b) + \log \sin \frac{1}{2}C = \log \cos \frac{1}{2}(A+B) + \log \cos \frac{1}{2}c,$

(iv) $\log \sin \frac{1}{2}(a+b) + \log \sin \frac{1}{2}C = \log \cos \frac{1}{2}(A-B) + \log \sin \frac{1}{2}c.$

651. THIRD METHOD. Based on Napier's Analogies (635), which, by taking the logarithm of each term, may be written thus:

(i) $\log \tan \frac{1}{2}(a-b) = \log \csc \frac{1}{2}(A+B) + \log \sin \frac{1}{2}(A-B) + \log \tan \frac{1}{2}c,$

(ii) $\log \tan \frac{1}{2}(a+b) = \log \sec \frac{1}{2}(A+B) + \log \cos \frac{1}{2}(A-B) + \log \tan \frac{1}{2}c,$

(iii) $\log \operatorname{ctn} \frac{1}{2}C = \log \csc \frac{1}{2}(a-b) + \log \sin \frac{1}{2}(a+b) + \log \tan \frac{1}{2}(A-B),$

(iv) $\log \operatorname{ctn} \frac{1}{2}C = \log \sec \frac{1}{2}(a-b) + \log \cos \frac{1}{2}(a+b) + \log \tan \frac{1}{2}(A+B).$

**652.** FOURTH METHOD. By drawing from B or from A a great-circle arc perpendicular to the opposite side, the required triangle is converted into the sum or the difference of two spherical right triangles.

| Perpendicular BQ. | Perpendicular AP. |
|---|---|
| $\operatorname{ctn} \text{ABQ} = \tan A \cos c,$ | $\operatorname{ctn} \text{BAP} = \tan B \cos c,$ |
| $\left.\begin{array}{l}\text{QBC} = B - \text{ABQ} \\ \text{or } \text{QBC} = 360^\circ + B - \text{ABQ}\end{array}\right\},$ | $\left.\begin{array}{l}\text{PAC} = A - \text{BAP} \\ \text{or } \text{PAC} = 360^\circ + A - \text{BAP}\end{array}\right\},$ |
| $\operatorname{ctn} a = \operatorname{ctn} c \sec \text{ABQ} \cos \text{QBC},$ | $\operatorname{ctn} b = \operatorname{ctn} c \sec \text{BAP} \cos \text{PAC},$ |
| $\cos C = \cos A \csc \text{ABQ} \sin \text{QBC}.$ | $\cos C = \cos B \csc \text{BAP} \sin \text{PAC}.$ |

**653.** CASE III. Given two sides and the angle opposite one of them $a$, $b$, and $A$, to find $B$, $C$, and $c$.

Two triangles,

| | |
|---|---|
| both possible when | $\sin a > \sin b \sin A,$ |
| both impossible when | $\sin a < \sin b \sin A,$ |
| identical when | $\sin a = \sin b \sin A.$ |

The value of the modulus is given by $M = \dfrac{\sin a}{\sin A}\cdot$ Hence the computed parts must be so taken as to give both triangles the *same modulus*, which will be positive or negative according to the given values of $a$ and $A$.

**654.** FIRST METHOD. By **602, 606,** and **607,**

$$\sin B = \frac{\sin b \sin A}{\sin a},$$

$$B_1 + B_2 = 180^\circ, \quad \text{or} \quad B_1 + B_2 = 540^\circ.$$

$$\sin c = \frac{\cos^2 a - \cos^2 b}{\cos a \sin b \cos A - \sin a \cos b \cos B},$$

$$\cos c = \frac{\sin b \cos b \cos A - \sin a \cos a \cos B}{\cos a \sin b \cos A - \sin a \cos b \cos B},$$

$$\sin C = \frac{\cos^2 A - \cos^2 B}{\cos A \sin B \cos a - \sin A \cos B \cos b},$$

$$\cos C = \frac{\sin A \cos A \cos b - \sin B \cos B \cos a}{\cos A \sin B \cos a - \sin A \cos B \cos b}.$$

Since the formulas give the values of both $\sin c$ and $\cos c$, there can result but one value of $c$ between $0^\circ$ and $360^\circ$ for each value of $B$ substituted. The same remark applies to $C$.

**655.** SECOND METHOD. The two values of $B$ having been computed as in the First Method, $c$ and $C$ are found by substituting the values of $B$ separately in Gauss's Equations (634), which may be written as follows:

$$\text{(i)}\ \log\cos\tfrac{1}{2}C - \log\cos\tfrac{1}{2}c = \log\sin\tfrac{1}{2}(A+B) - \log\cos\tfrac{1}{2}(a-b),$$

$$\text{(ii)}\ \log\cos\tfrac{1}{2}C - \log\sin\tfrac{1}{2}c = \log\sin\tfrac{1}{2}(A-B) - \log\sin\tfrac{1}{2}(a-b),$$

$$\text{(iii)}\ \log\sin\tfrac{1}{2}C - \log\cos\tfrac{1}{2}c = \log\cos\tfrac{1}{2}(A+B) - \log\cos\tfrac{1}{2}(a+b),$$

$$\text{(iv)}\ \log\sin\tfrac{1}{2}C - \log\sin\tfrac{1}{2}c = \log\cos\tfrac{1}{2}(A-B) - \log\sin\tfrac{1}{2}(a+b).$$

**656.** THIRD METHOD. The two values of $B$, having been computed as in the First Method, are separately substituted in Napier's Analogies (635), which may be written thus:

$$\text{(i)}\ \log\tan\tfrac{1}{2}c = \log\sin\tfrac{1}{2}(A+B) + \log\csc\tfrac{1}{2}(A-B) + \log\tan\tfrac{1}{2}(a-b),$$

$$\text{(ii)}\ \log\tan\tfrac{1}{2}c = \log\cos\tfrac{1}{2}(A+B) + \log\sec\tfrac{1}{2}(A-B) + \log\tan\tfrac{1}{2}(a+b),$$

$$\text{(iii)}\ \log\operatorname{ctn}\tfrac{1}{2}C = \log\sin\tfrac{1}{2}(a+b) + \log\csc\tfrac{1}{2}(a-b) + \log\tan\tfrac{1}{2}(A-B),$$

$$\text{(iv)}\ \log\operatorname{ctn}\tfrac{1}{2}C = \log\cos\tfrac{1}{2}(a+b) + \log\sec\tfrac{1}{2}(a-b) + \log\tan\tfrac{1}{2}(A+B).$$

**657.** FOURTH METHOD. By drawing a great-circle arc CR from C perpendicular to the opposite side, two right triangles CRA and CRB are formed which may be solved as follows:

$$\begin{aligned}
\sin B &= \csc a \sin b \sin A,\\
\tan \mathrm{AR} &= \tan b \cos A,\\
\cos \mathrm{RB} &= \cos \mathrm{AR} \cos a \sec b,\\
&\left.\begin{aligned} c_1 &= \mathrm{AR} + \mathrm{RB}\\ c_2 &= \mathrm{AR} - \mathrm{RB}\end{aligned}\right\},\\
\tan \mathrm{ACR} &= \sec b \operatorname{ctn} A,\\
\cos \mathrm{RCB} &= \cos \mathrm{ACR} \operatorname{ctn} a \tan b,\\
&\left.\begin{aligned} C_1 &= \mathrm{ACR} + \mathrm{RCB}\\ C_2 &= \mathrm{ACR} - \mathrm{RCB}\end{aligned}\right\}.
\end{aligned}$$

**658.** CASE IV. Given two angles and the side opposite one of them, $A$, $B$, and $a$, to find $b$, $c$, and $C$.

Two triangles,

| | |
|---|---|
| both possible when | $\sin A > \sin B \sin a,$ |
| both impossible when | $\sin A < \sin B \sin a,$ |
| identical when | $\sin A = \sin B \sin a.$ |

The value of the modulus is given by $M = \dfrac{\sin a}{\sin A}$. Hence the computed parts must be so taken as to give both triangles the *same modulus*, which will be positive or negative according to the given values of $a$ and $A$.

**659.** FIRST METHOD. By **602, 606,** and **607,**

$$\sin b = \frac{\sin B \sin a}{\sin A},$$

$$b_1 + b_2 = 180°, \quad \text{or} \quad b_1 + b_2 = 540°.$$

$$\sin c = \frac{\cos^2 a - \cos^2 b}{\cos a \sin b \cos A - \sin a \cos b \cos B},$$

$$\cos c = \frac{\sin b \cos b \cos A - \sin a \cos a \cos B}{\cos a \sin b \cos A - \sin a \cos b \cos B},$$

$$\sin C = \frac{\cos^2 A - \cos^2 B}{\cos A \sin B \cos a - \sin A \cos B \cos b},$$

$$\cos C = \frac{\sin A \cos A \cos b - \sin B \cos B \cos a}{\cos A \sin B \cos a - \sin A \cos B \cos b}.$$

For each value of $b$ substituted in the last four equations, there results but one value of $c$ and one value of $C$ between 0° and 360°.

**660.** SECOND METHOD. The two values of $b$ having been computed as in the First Method, $c$ and $C$ are found by substituting the values of $b$ separately in Gauss's Equations, which for this purpose are written in the same form as in **655.**

**661.** THIRD METHOD. The two values of $b$, having been computed as in the First Method, are separately substituted in Napier's Analogies, which are written in the same form as in **656.**

**662.** FOURTH METHOD. By drawing a great-circle arc CR from C perpendicular to the opposite side, two right triangles are formed, which may be solved as follows:

$$\sin b = \csc A \sin B \sin a,$$
$$\operatorname{ctn} \text{BCR} = \tan B \cos a,$$
$$\sin \text{RCA} = \sin \text{BCR} \cos A \sec B,$$

which gives supplementary values of RCA,

$$\left.\begin{aligned} C_1 &= \text{BCR} + \text{RCA}_1 \\ C_2 &= \text{BCR} + \text{RCA}_2 \end{aligned}\right\}.$$

$$\operatorname{ctn} \text{BR} = \sec B \operatorname{ctn} a,$$
$$\sin \text{RA} = \sin \text{BR} \operatorname{ctn} A \tan B,$$

which gives supplementary values of RA.

$$\left.\begin{aligned} c_1 &= \text{BR} + \text{RA}_1 \\ c_2 &= \text{BR} + \text{RA}_2 \end{aligned}\right\}.$$

If the values of sin RCA and sin RA are negative, then

$$\text{RCA}_1 + \text{RCA}_2 = 540° \text{ and } \text{RA}_1 + \text{RA}_2 = 540°.$$

**663.** CASE V. Given the three sides, $a$, $b$, and $c$, to find the three angles $A$, $B$, and $C$.

Two triangles always possible.

$$A_1 + A_2 = B_1 + B_2 = C_1 + C_2 = 360°.$$

Which of the computed values of the angles belong to one triangle and which to the other is decided by means of the equations

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} = M.$$

One triangle has a positive, the other a negative modulus.

**664.** FIRST METHOD. By **602**,

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c},$$

$$\cos B = \frac{\cos b - \cos c \cos a}{\sin c \sin a},$$

$$\cos C = \frac{\cos c - \cos a \cos b}{\sin a \sin b}.$$

**665.** Second Method. The halves of the angles can be computed from their sines (**611**), or from their cosines (**612**), or from their tangents (**613**). The most convenient form of computation from the tangents is the following derived from **620** and **622**:

$$\tan r = \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}},$$

$$\tan \tfrac{1}{2}A = \frac{\tan r}{\sin(s-a)},\quad \tan \tfrac{1}{2}B = \frac{\tan r}{\sin(s-b)},\quad \tan \tfrac{1}{2}C = \frac{\tan r}{\sin(s-c)},$$

The right-hand members of these equations, being square roots, are to be taken as both positive and negative. Hence each half angle has two values, which doubled are the two values of the whole angle.

**666.** Case VI. Given the three angles $A$, $B$, and $C$, to find the three sides, $a$, $b$, and $c$.

Two triangles always possible.

$$a_1 + a_2 = b_1 + b_2 = c_1 + c_2 = 360°.$$

Which of the computed values of the sides belong to one triangle and which to the other is decided by means of the equations

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} = M.$$

One triangle has a positive, the other a negative modulus.

**667.** First Method. By **604**,

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C},$$

$$\cos b = \frac{\cos B + \cos C \cos A}{\sin C \sin A},$$

$$\cos c = \frac{\cos C + \cos A \cos B}{\sin A \sin B}.$$

**668.** Second Method. The halves of the sides can be computed from their sines (**617**), or from their cosines (**616**), or from their cotangents (**618**). The most conven-

ient form of computation from the cotangents is the following derived from **624** and **625**:

$$\operatorname{ctn} R = \sqrt{\frac{\cos(S-A)\cos(S-B)\cos(S-C)}{-\cos S}},$$

$$\operatorname{ctn}\tfrac{1}{2}a = \frac{\operatorname{ctn} R}{\cos(S-A)},\ \operatorname{ctn}\tfrac{1}{2}b = \frac{\operatorname{ctn} R}{\cos(S-B)},\ \operatorname{ctn}\tfrac{1}{2}c = \frac{\operatorname{ctn} R}{\cos(S-C)}.$$

The right-hand members of these equations, being square roots, are to be taken as both positive and negative. Hence each half side has two values, which doubled are the two values of the whole side.

*Special Formulas for Spherical Right Triangles.*

For more accurate results than the general formulas give in extreme cases, the following equations may be used.

See also **676–681**.

When $A$ is near 90°,

**669.** $$\tan(45° - \tfrac{1}{2}A) = \sqrt{\frac{\tan\frac{1}{2}(h-a)}{\tan\frac{1}{2}(h+a)}} = \sqrt{\tan\tfrac{1}{2}(B+b)\tan\tfrac{1}{2}(B-b)}.$$

When $A$ is near 0° or near 180°,

**670.** $$\tan\tfrac{1}{2}A = \sqrt{\frac{\sin(h-b)}{\sin(h+b)}}.$$

When $a$ is near 90°,

**671.** $$\tan(45° - \tfrac{1}{2}a) = \sqrt{\frac{\sin(B-b)}{\sin(B+b)}}.$$

When $a$ is near 0° or near 180°,

**672.** $$\tan\tfrac{1}{2}a = \sqrt{\tan\tfrac{1}{2}(h+b)\tan\tfrac{1}{2}(h-b)}.$$

When $h$ is near 90°,

**673.** $$\tan(45° - \tfrac{1}{2}h) = \sqrt{\frac{\tan\frac{1}{2}(A-a)}{\tan\frac{1}{2}(A+a)}}.$$

When $h$ is near 0°,

**674.** $$\sin\tfrac{1}{2}h = \sqrt{\frac{-\cos(A+B)}{2\sin A\sin B}}.$$

When $h$ is near 180°,

**675.** $$\cos\tfrac{1}{2}h = \sqrt{\frac{\cos(A-B)}{2\sin A\sin B}}.$$

## *Accurate Computation of Angles near* 0° *and near* 90°.

**676.** *Values of* S *and* T.

$x''$ = the angle expressed in seconds.
$(90° - x)''$ = the complement of the angle expressed in seconds.

$S = \log\left(\frac{\sin x}{x''}\right)$.

| $x''$. | S. | log sin $x$. |
|---|---|---|
| 0 | | |
| | 4.68 557 | |
| 2409 | | 8.06 740 |
| | 4.68 556 | |
| 3417 | | 8.21 920 |
| | 4.68 555 | |
| 4190 | | 8.30 776 |
| | 4.68 554 | |
| 4840 | | 8.37 038 |
| | 4.68 553 | |
| 5414 | | 8.41 904 |
| | 4.68 552 | |
| 5932 | | 8.45 872 |
| | 4.68 551 | |
| 6408 | | 8.49 223 |
| | 4.68 550 | |
| 6851 | | 8.52 125 |
| | 4.68 549 | |
| 7206 | | 8.54 321 |

1. When the angle $x$ is between 0° and 2°,
   $\log \sin x = \log x'' + S.$
   $\log \tan x = \log x'' + T.$
   $\log \operatorname{ctn} x = 20 - \log \tan x.$
2. When the angle $x$ is between 88° and 90°,
   $\log \cos x = \log (90° - x)'' + S.$
   $\log \operatorname{ctn} x = \log (90° - x)'' + T.$
   $\log \tan x = 20 - \log \operatorname{ctn} x.$
3. When the logarithm of the function is less than 8.54321,
   $\log x'' = \log \sin x - S.$
   $\log x'' = \log \tan x - T.$
   $\log (90° - x)'' = \log \cos x - S.$
   $\log (90° - x)'' = \log \operatorname{ctn} x - T.$
4. When the logarithm of the function is greater than 11.45678,
   $\log x'' = 20 - \log \operatorname{ctn} x - T.$
   $\log (90° - x)'' = 20 - \log \tan x - T.$

$T = \log\left(\frac{\tan x}{x''}\right)$.

| $x''$. | T. | log tan $x$. |
|---|---|---|
| 0 | | |
| | 4.68 557 | |
| 200 | | 6.98 660 |
| | 4.68 558 | |
| 1726 | | 7.92 263 |
| | 4.68 559 | |
| 2432 | | 8.07 156 |
| | 4.68 560 | |
| 2976 | | 8.15 924 |
| | 4.68 561 | |
| 3434 | | 8.22 142 |
| | 4.68 562 | |
| 3838 | | 8.26 973 |
| | 4.68 563 | |
| 4204 | | 8.30 930 |
| | 4.68 564 | |
| 4540 | | 8.34 270 |
| | 4.68 565 | |
| 4853 | | 8.37 167 |
| | 4.68 566 | |
| 5146 | | 8.39 713 |
| | 4.68 567 | |
| 5424 | | 8.41 999 |
| | 4.68 568 | |
| 5689 | | 8.44 072 |
| | 4.68 569 | |
| 5941 | | 8.45 955 |
| | 4.68 570 | |
| 6184 | | 8.47 697 |
| | 4.68 571 | |
| 6417 | | 8.49 305 |
| | 4.68 572 | |
| 6642 | | 8.50 802 |
| | 4.68 573 | |
| 6859 | | 8.52 200 |
| | 4.68 574 | |
| 7070 | | 8.53 516 |
| | 4.68 575 | |
| 7202 | | 8.54 321 |

When tables of the values of S and T are not within reach, the following equations may be used to compute very closely any needed values:

677. $$\begin{cases} S = 4.6855749 - \frac{1}{3}\,(10 - \log\cos x) \\ T = 4.6855749 + \frac{2}{3}\,(10 - \log\cos x) \end{cases}$$

Or, the formulas may be written as follows:

678. $\log\sin x = \log x'' + 4.6855749 - \frac{1}{3}\,(10 - \log\cos x).$

679. $\log\tan x = \log x'' + 4.6855749 + \frac{2}{3}\,(10 - \log\cos x).$

680. $\log x'' = \log\sin x + 5.3144251 + \frac{1}{3}\,(10 - \log\cos x) - 10.$

681. $\log x'' = \log\tan x + 5.3144251 - \frac{2}{3}\,(10 - \log\cos x) - 10.$

# SECTION III.

## HYPERBOLIC FUNCTIONS.

*Definitions.*

Connected with the equilateral hyperbola are six functions which, from their close resemblance to the six trigonometric or circular functions, have been named the hyperbolic sine, the hyperbolic cosine, and so on. They are derived geometrically from certain properties of central conics, or they may be defined algebraically through their relations to the exponential $e^u$. Thus in the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

OA being the major semi-axis, OP any radius vector, and $u$ double the measure of the hyperbolic sector AOP, the hyperbolic sine and cosine are defined by the equations

**701.** $$\begin{cases} \text{hyp. sine} \quad \text{of } u = \dfrac{y}{b}, \\ \text{hyp. cosine of } u = \dfrac{x}{a}. \end{cases}$$

The quotient of these is defined as the hyperbolic tangent. The hyperbolic cotangent, secant, and cosecant are defined as the reciprocals of the hyperbolic tangent, cosine, and sine respectively.

Or, the defining equations may be

**702.** $$\begin{cases} \text{hyp. cosine of } u = \tfrac{1}{2}(e^u + e^{-u}) \\ \text{hyp. sine} \quad \text{of } u = \tfrac{1}{2}(e^u - e^{-u}) \\ \quad \text{etc.} \qquad\qquad \text{etc.} \end{cases}$$

which are the analogues of

$$\text{circular } \cos u = \tfrac{1}{2}(e^{iu} + e^{-iu})$$

$$\text{circular } \sin u = \frac{1}{2i}(e^{iu} - e^{-iu}).$$

$$\text{etc.} \qquad\qquad \text{etc.}$$

As to notation, it is convenient to use the abbreviations of the names of the six circular functions, sin, cos, tan, ctn, sec, csc, marking the distinction by using a capital initial letter and adding the letter $h$; thus,

Sinh, Cosh, Tanh, Ctnh, Sech, Csch.*

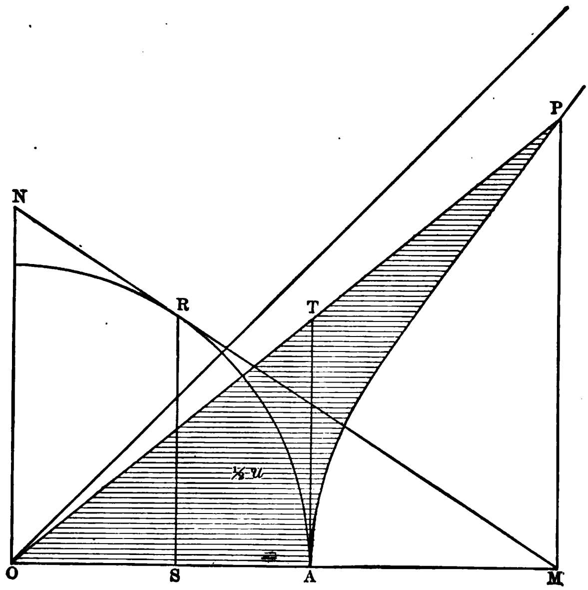


FIGURE 2.

Hyperbolic Sector AOP $= \frac{1}{2}u$.
Circular Sector AOR $= \frac{1}{2}\theta$.
Radius OA $= 1$.

In the *equilateral hyperbola*, $x^2 - y^2 = 1$, Figure 2, the hyperbolic functions of $u$ (double the area of the sector AOP) are represented by, and are numerically equal to the lines drawn as follows:

703. $$\begin{cases} \text{Sinh } u = \text{the ordinate of P} = \text{MP}, \\ \text{Cosh } u = \text{the abscissa of P} = \text{OM}, \\ \text{Tanh } u = \dfrac{\text{MP}}{\text{OM}} = \dfrac{\text{AT}}{\text{OA}} = \dfrac{\text{AT}}{1} = \text{AT}. \end{cases}$$

* Some writers merely add the $h$, but the capital initial serves well to catch the attention.

If, in the integral for the area of a segment of an equilateral hyperbola

$$\text{AMP} = \int^{x}(x^2 - a^2)^{\frac{1}{2}}\,dx = \frac{x}{2}(x^2 - a^2)^{\frac{1}{2}} - \frac{a^2}{2}\,\text{Cosh}^{-1}\frac{x}{a},$$

we write $a$ Cosh $u$ for $x$, the result is

$$\text{AMP} = a^2\int^{u}\text{Sinh}^2 u.du = \frac{a^2}{2}\int^{u}(\text{Cosh } 2u - 1)\,du$$

$$= \frac{a^2}{4}(\text{Sinh } 2u - 2u)$$

$$= \frac{a^2}{2}(\text{Sinh } u \text{ Cosh } u - u)$$

$$= \text{OMP} - \text{OAP}, \text{ where } \text{OA} = a.$$

From the equation $x^2 - y^2 = 1$ and **703** it follows directly that

**704.** $$\text{Cosh}^2 u - \text{Sinh}^2 u = 1.$$

This equation and the four defining equations

**705.** $\text{Tanh}\,u = \dfrac{\text{Sinh } u}{\text{Cosh } u}$, **706.** $\text{Ctnh}\,u = \dfrac{\text{Cosh } u}{\text{Sinh } u}$,

**707.** $\text{Sech } u = \dfrac{1}{\text{Cosh } u}$, **708.** $\text{Csch } u = \dfrac{1}{\text{Sinh } u}$,

make it possible, when the numerical value of one function is given, to compute the values of the other five.

*Relations of Hyperbolic Functions to one another.*

From the equations **704–708** are deduced

**709.** $$1 - \text{Tanh}^2 u = \text{Sech}^2 u.$$

**710.** $$\text{Ctnh}^2 u - 1 = \text{Csch}^2 u.$$

**711.** $$\text{Sinh } u = \sqrt{\text{Cosh}^2 u - 1} = \frac{\text{Tanh } u}{\sqrt{1 - \text{Tanh}^2 u}}$$

$$= \frac{1}{\sqrt{\text{Ctnh}^2 u - 1}} = \frac{\sqrt{1 - \text{Sech}^2 u}}{\text{Sech } u} = \frac{1}{\text{Csch } u}.$$

712. $$\operatorname{Cosh} u = \sqrt{\operatorname{Sinh}^2 u + 1} = \frac{1}{\sqrt{1 - \operatorname{Tanh}^2 u}},$$

$$= \frac{\operatorname{Ctnh} u}{\sqrt{\operatorname{Ctnh}^2 u - 1}} = \frac{\sqrt{\operatorname{Csch}^2 u + 1}}{\operatorname{Csch} u} = \frac{1}{\operatorname{Sech} u}.$$

713. $$\operatorname{Tanh} u = \frac{\operatorname{Sinh} u}{\sqrt{\operatorname{Sinh}^2 u + 1}} = \frac{\sqrt{\operatorname{Cosh}^2 u - 1}}{\operatorname{Cosh} u},$$

$$= \sqrt{1 - \operatorname{Sech}^2 u} = \frac{1}{\sqrt{\operatorname{Csch}^2 u + 1}} = \frac{1}{\operatorname{Ctnh} u}.$$

*Relations between the hyperbolic and the circular functions of the same variable.*

In the exponential equivalents of the circular functions

$$\cos x = \tfrac{1}{2}(e^{ix} + e^{-ix})$$
$$i \sin x = \tfrac{1}{2}(e^{ix} - e^{-ix})$$

and in those of the hyperbolic functions

$$\operatorname{Cosh} x = \tfrac{1}{2}(e^{x} + e^{-x})$$
$$\operatorname{Sinh} x = \tfrac{1}{2}(e^{x} - e^{-x}),$$

by writing $iu$ in place of $x$, there are found the following relations:

714. $$\begin{cases} \sin iu = i \operatorname{Sinh} u, & \operatorname{Sinh} iu = i \sin u, \\ \cos iu = \operatorname{Cosh} u, & \operatorname{Cosh} iu = \cos u, \\ \tan iu = i \operatorname{Tanh} u, & \operatorname{Tanh} iu = i \tan u, \\ \operatorname{ctn} iu = -i \operatorname{Ctnh} u, & \operatorname{Ctnh} iu = -i \operatorname{ctn} u, \\ \sec iu = \operatorname{Sech} u, & \operatorname{Sech} iu = \sec u, \\ \csc iu = -i \operatorname{Csch} u, & \operatorname{Csch} iu = -i \csc u. \end{cases}$$

715. $$\begin{cases} \cos y + i \sin y = \operatorname{Cosh} iy + \operatorname{Sinh} iy = e^{iy}, \\ \cos y - i \sin y = \operatorname{Cosh} iy - \operatorname{Sinh} iy = e^{-iy}, \\ \cos iy + i \sin iy = \operatorname{Cosh} y - \operatorname{Sinh} y = e^{-y}, \\ \cos iy - i \sin iy = \operatorname{Cosh} y + \operatorname{Sinh} y = e^{y}. \end{cases}$$

*Hyperbolic functions of $-u$.*

716. $$\begin{cases} \operatorname{Sinh}(-u) = -\operatorname{Sinh} u, & \operatorname{Ctnh}(-u) = -\operatorname{Ctnh} u, \\ \operatorname{Cosh}(-u) = +\operatorname{Cosh} u, & \operatorname{Sech}(-u) = +\operatorname{Sech} u, \\ \operatorname{Tanh}(-u) = -\operatorname{Tanh} u, & \operatorname{Csch}(-u) = -\operatorname{Csch} u. \end{cases}$$

*Variations and Cardinal Values.*

As the radius vector OP swings from the axis OA to the upper right-hand asymptote, the variable, $u$, passes through all positive values from 0 to $+\infty$. At the same time $\operatorname{Sinh} u$ passes from 0 to $+\infty$; $\operatorname{Cosh} u$ from $+1$ to $+\infty$, and $\operatorname{Tanh} u$ from 0 to $+1$. On the other hand, as OP swings from the axis OA to the lower right-hand asymptote, $u$ passes through all negative values from 0 to $-\infty$. $\operatorname{Sinh} u$ is likewise negative, passing from 0 to $-\infty$; $\operatorname{Cosh} u$ is positive, passing from $+1$ to $+\infty$; and $\operatorname{Tanh} u$ is negative, passing from 0 to $-1$.

For all real values of $u$, therefore,

**717.** $$\begin{cases} -\infty \overline{\gtrless} \operatorname{Sinh} u \leqq +\infty, \\ +1 \overline{\gtrless} \operatorname{Cosh} u \leqq +\infty, \\ -1 \overline{\gtrless} \operatorname{Tanh} u \leqq +1. \end{cases}$$

Cardinal values,

**718.**

$$\begin{cases} \operatorname{Sinh} 0 = 0, & \operatorname{Cosh} 0 = 1, & \operatorname{Tanh} 0 = 0, \\ \operatorname{Sinh} \infty = \infty, & \operatorname{Cosh} \infty = \infty, & \operatorname{Tanh} \infty = 1, \\ \operatorname{Sinh}(-\infty) = -\infty, & \operatorname{Cosh}(-\infty) = \infty, & \operatorname{Tanh}(-\infty) = -1. \end{cases}$$

*Relations of Hyperbolic to Trigonometric Formulas.*

By means of the relations expressed in **714**, any general formula of trigonometry can be changed into an analogous formula involving the hyperbolic functions. Thus, by putting $x = iu$ and reducing,

$\cos^2 x + \sin^2 x = 1$ becomes $\operatorname{Cosh}^2 u - \operatorname{Sinh}^2 u = 1$.
$1 + \tan^2 x = \sec^2 x$ becomes $1 - \operatorname{Tanh}^2 u = \operatorname{Sech}^2 u$.
$\operatorname{ctn}^2 x + 1 = \csc^2 x$ becomes $\operatorname{Ctnh}^2 u - 1 = \operatorname{Csch}^2 u$.

*The Addition and Subtraction Formulas.*

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y,$$

by putting $x = iu$ and $y = iv$, become

**719.** $\operatorname{Sinh}(u \pm v) = \operatorname{Sinh} u \operatorname{Cosh} v \pm \operatorname{Cosh} u \operatorname{Sinh} v$.
**720.** $\operatorname{Cosh}(u \pm v) = \operatorname{Cosh} u \operatorname{Cosh} v \pm \operatorname{Sinh} u \operatorname{Sinh} v$.

From these are deduced

721. $\text{Sinh}\, u + \text{Sinh}\, v = 2\,\text{Sinh}\, \tfrac{1}{2}(u+v)\,\text{Cosh}\, \tfrac{1}{2}(u-v).$

722. $\text{Sinh}\, u - \text{Sinh}\, v = 2\,\text{Cosh}\, \tfrac{1}{2}(u+v)\,\text{Sinh}\, \tfrac{1}{2}(u-v).$

723. $\text{Cosh}\, u + \text{Cosh}\, v = 2\,\text{Cosh}\, \tfrac{1}{2}(u+v)\,\text{Cosh}\, \tfrac{1}{2}(u-v).$

724. $\text{Cosh}\, u - \text{Cosh}\, v = 2\,\text{Sinh}\, \tfrac{1}{2}(u+v)\,\text{Sinh}\, \tfrac{1}{2}(u-v).$

725. $\text{Sinh}\, 2u = 2\,\text{Sinh}\, u\,\text{Cosh}\, u.$

726. $$\begin{aligned}\text{Cosh}\, 2u &= \text{Cosh}^2 u + \text{Sinh}^2 u,\\ &= 2\,\text{Sinh}^2 u + 1,\\ &= 2\,\text{Cosh}^2 u - 1.\end{aligned}$$

727. $$\text{Sinh}\, 2u = \frac{2\,\text{Tanh}\, u}{1-\text{Tanh}^2 u}.$$

728. $$\text{Cosh}\, 2u = \frac{1+\text{Tanh}^2 u}{1-\text{Tanh}^2 u}.$$

729. $$\text{Tanh}\, 2u = \frac{2\,\text{Tanh}\, u}{1+\text{Tanh}^2 u}.$$

730. $$\text{Sinh}\,\frac{u}{2} = \sqrt{\tfrac{1}{2}(\text{Cosh}\, u - 1)}.$$

731. $$\text{Cosh}\,\frac{u}{2} = \sqrt{\tfrac{1}{2}(\text{Cosh}\, u + 1)}.$$

732. $$\text{Tanh}\frac{u}{2} = \sqrt{\frac{\text{Cosh}\, u - 1}{\text{Cosh}\, u + 1}} = \frac{\text{Cosh}\, u - 1}{\text{Sinh}\, u} = \frac{\text{Sinh}\, u}{\text{Cosh}\, u + 1}.$$

733. $\text{Sinh}^{-1} x \pm \text{Sinh}^{-1} y = \text{Sinh}^{-1}\left[x\sqrt{1+y^2} \pm y\sqrt{1+x^2}\right].$

734. $\text{Cosh}^{-1} x \pm \text{Cosh}^{-1} y = \text{Cosh}^{-1}\left[xy \pm \sqrt{x^2-1}\,\sqrt{y^2-1}\right].$

*Hyperbolic Functions of a Complex Variable, $u = x + iy$.*

735. $\text{Sinh}\,(x+iy) = \text{Sinh}\, x \cos y + i\,\text{Cosh}\, x \sin y.$

736. $\text{Cosh}\,(x+iy) = \text{Cosh}\, x \cos y + i\,\text{Sinh}\, x \sin y.$

737. $\text{Sinh}\,(x \pm iy) = \pm i \sin(y \mp ix).$

738. $\text{Cosh}\,(x \pm iy) = \cos(y \mp ix).$

739. $\sin(x \pm iy) = \pm i\,\text{Sinh}\,(y \mp ix).$

740. $\cos(x \pm iy) = \text{Cosh}\,(y \mp ix).$

*Periodicity.*

The hyperbolic functions, like the circular, are periodic. The period of Sinh $u$ and of Cosh $u$ is $2\pi i$; that of Tanh $u$ and of Ctnh $u$ is $\pi i$. Thus,

741. $\text{Sinh}\,(u + 2\pi i) = \text{Sinh}\,u.$

742. $\text{Cosh}\,(u + 2\pi i) = \text{Cosh}\,u.$

743. $\text{Sinh}\,k\pi i = 0.$

744. $\text{Cosh}\,k\pi i = (-1)^k.$

745. $\text{Tanh}\,k\pi i = 0.$

746. $\text{Sinh}\,(u + \pi i) = -\text{Sinh}\,u.$

747. $\text{Cosh}\,(u + \pi i) = -\text{Cosh}\,u.$

748. $\text{Tanh}\,(u + \pi i) = \text{Tanh}\,u.$

749. $\text{Ctnh}\,(u + \pi i) = \text{Ctnh}\,u.$

750. $\text{Sinh}\,(2k + 1)\,\frac{1}{2}\pi i = \pm i.$

751. $\text{Cosh}\,(2k + 1)\frac{1}{2}\pi i = 0.$

752. $\text{Sinh}\,(u + \frac{1}{2}\pi i) = i\,\text{Cosh}\,u.$

753. $\text{Cosh}\,(u + \frac{1}{2}\pi i) = i\,\text{Sinh}\,u.$

*Hyperbolic Anti-Functions expressed as Logarithms.*

754. $\text{Sinh}^{-1}\,u = \log_e(u + \sqrt{u^2 + 1}).$

755. $\text{Cosh}^{-1}\,u = \log_e(u + \sqrt{u^2 - 1}).$

756. $\text{Tanh}^{-1}\,u = \frac{1}{2}\log_e\dfrac{1 + u}{1 - u}.$

757. $\text{Ctnh}^{-1}\,u = \frac{1}{2}\log_e\dfrac{u + 1}{u - 1}.$

758. $\text{Sech}^{-1}\,u = \log_e\dfrac{1 + \sqrt{1 - u^2}}{u}.$

759. $\text{Csch}^{-1}\,u = \log_e\dfrac{1 + \sqrt{1 + u^2}}{u}.$

*The Gudermanian Function and Angle.*

If a straight line, Figure 2, be drawn from the foot of the ordinate of P tangent to the circle which has the axis of the equilateral hyperbola for its diameter its length

760. $\text{MR} = \sqrt{\overline{\text{OM}}^2 - \overline{\text{OR}}^2} = \sqrt{\text{Cosh}^2 u - 1} = \text{Sinh}\, u = \text{MP}.$

The angle AOR is the Gudermanian Angle. Its radian measure, $\theta$, is numerically equal to double the area of the circular sector AOR, the radius OA being unity. The circular sector AOR and the hyperbolic sector AOP are functions the one of the other. If $\theta$ and $u$ are double their respective measures, then $\theta$ is the gudermanian of $u$ and $u$ is the anti-gudermanian of $\theta$; and the relation is expressed by

761. $\theta = gdu, \qquad u = gd^{-1}\,\theta.$

The six hyperbolic functions of $u$ severally have equivalents among the six circular functions of the Gudermanian Angle, thus:

762. $$\begin{cases} \text{Cosh}\, u = \sec\,\theta = \text{OM} \\ \text{Sinh}\, u = \tan\,\theta = \text{MP} = \text{RM} \\ \text{Tanh}\, u = \sin\,\theta = \text{AT} = \text{SR} \\ \text{Ctnh}\, u = \csc\,\theta = \text{ON} \\ \text{Sech}\, u = \cos\,\theta = \text{OS} \\ \text{Csch}\, u = \text{ctn}\,\theta = \text{RN} \end{cases}$$

Also,

763. $\text{Tanh}\,\tfrac{1}{2}u = \tan\,\tfrac{1}{2}\theta.$

764. $gd\,0 = 0,\ gd\,\infty = \tfrac{1}{2}\pi,\ gd\,(-\infty) = -\tfrac{1}{2}\pi.$

765. $u = gd^{-1}\,\theta = \log_e \tan\,(\tfrac{1}{4}\pi + \tfrac{1}{2}\theta).$

# SECTION IV.

## DIFFERENTIAL AND INTEGRAL CALCULUS.

### *Limits.*

**801.** $\lim\limits_{x \doteq \infty} \left(\frac{a}{x} + b\right) = b.$*

**802.** $\lim\limits_{\omega \doteq \infty} \left(1 + \frac{1}{\omega}\right)^{\omega} = e.$

**803.** $\lim\limits_{\partial \doteq 0} \left(1 + \partial\right)^{\frac{1}{\partial}} = e.$

**804.** $\lim\limits_{\partial \doteq 0} \frac{\log(1 + \partial)}{\partial} = \log e.$

**805.** $\lim\limits_{\theta \doteq 0} \frac{a^{\theta} - 1}{\theta} = \frac{1}{\log_a e} = \log_e a.$

**806.** $\lim\limits_{\partial \doteq 0} \frac{(1 + \partial)^m - 1}{\partial} = m.$

**807.** $\lim\limits_{\theta \doteq 0} \frac{\sin\theta}{\theta} = 1.$ $\quad \lim \frac{\tan\theta}{\theta} = 1.$

### *Definitions and Notation.*

**808.** If $y = f(x)$, then $y + \Delta y = f(x + \Delta x)$.

Here $\Delta x$ and $\Delta y$ denote the finite differences, or increments, of the independent variable $x$ and the dependent variable $y$, respectively.

The value of $\Delta x$ is arbitrary (may be any we please) and in particular may be made to vary and to approach 0 as nearly as we please. The value of $\Delta y$ depends on that of $\Delta x$. If $\Delta x$ approaches 0, $\Delta y$ does the same, and both reach their limit 0 at the same time.

* The sign $\doteq$ is read " approaches infinitely near to."

The definite value, or *limit*, which the ratio

809. $$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

approaches when $\Delta x$ and $\Delta y$ simultaneously approach 0 is the *differential coefficient* of the function with respect to $x$.

It is denoted by $$\frac{dy}{dx} \text{ or } \frac{d}{dx} f(x).$$

The differential coefficient is also called *differential quotient, derivative*, or *derived function.* It is also denoted by

$$D_x y,\ D_x f(x),\ \text{or } f'(x).$$

The symbols $\frac{d}{dx}$ and $D_x$ are to be regarded as symbols of operation, that of differentiation, upon $f(x)$; the result of which operation is a new function of $x$, often denoted by $f'(x)$.

The symbols $\frac{d}{dx}$ and $D_x$, $\frac{d}{dy}$ and $D_y$, etc., when applied to functions $u$, $v$, etc., are obedient to

The Distributive Law,

810. $$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx},$$

$$\text{or } D_x(u + v) = D_x u + D_x v.$$

The Commutative Law.

811. $$\frac{d}{dx}\left(\frac{du}{dy}\right) = \frac{d}{dy}\left(\frac{du}{dx}\right), \text{ or } D_x(D_y u) = D_y(D_x u).$$

The Law of Indices,

812. $$\frac{d^m}{dx^m}\frac{d^n}{dx^n}u = \frac{d^{m+n}}{dx^{m+n}}u, \text{ or } D_x^m D_x^n u = D_x^{m+n} u.$$

The symbols $dx$ and $df(x)$ denote the *differentials* of $x$ and $f(x)$ respectively. They express the *rates* at which $x$ and $f(x)$ are changing their values at any given instant,

or, what is the same thing, for any given value of the independent variable. They have, in general, finite values, and are connected with each other and with the corresponding differential coefficients by the equations

**813.** $df(x) = \frac{d}{dx} f(x).dx = D_x f(x).dx = f'(x).dx.$

The independent variable, $x$, is generally assumed to change at a *uniform rate*, that is, it is *equicrescent*; and therefore $dx$ is constant and $d(dx) = 0$. But if $x$ be not equicrescent, as when it is a function of some other variable, then $dx$ is not a constant, and $d(dx)$ is not 0.

The process of integration is the inverse of that of differentiation. A given function is regarded as the differential coefficient or derivative of some primative function, and this primative function is to be found.

The symbol for integration is $\int$.

Its effect is to reverse or undo the effect of $d$.

**814.** Thus $\left\{\begin{array}{l} df(x) = f'(x).dx. \\ \int f'(x).dx = f(x). \end{array}\right.$

*Fundamental Formulas.*

In the following sixteen formulas $u, v, w, \ldots$ are functions of some independent variable, $x$ or $t$.

**815.** $d(au) = adu.$ **816.** $\int adu = a\int du = au + C.$

**817.** $d(u + v + w + \ldots) = du + dv + dw + \ldots$

**818.** $\int (u + v + \ldots) dx = \int udx + \int vdx + \ldots$

**819.** $d(uv) = vdu + udv.$ **820.** $\int udv = uv - \int vdu.$

**821.** $\frac{d(uv)}{uv} = \frac{du}{u} + \frac{dv}{v}.$

822. $\dfrac{d\,(uvw\ldots)}{uvw\ldots} = \dfrac{du}{u} + \dfrac{dv}{v} + \dfrac{dw}{w} + \ldots$

823. $d\left(\dfrac{u}{v}\right) = \dfrac{vdu - udv}{v^2}.$

824. $d\log_a u = \log_a e.\dfrac{du}{u}.$

825. $d\log_e u = \dfrac{du}{u}.$ 826. $\displaystyle\int\frac{du}{u} = \log_e u + C.$

827. $da^u = \log_e a.a^u.du.$

828. $de^u = e^u du.$ 829. $\displaystyle\int e^u du = e^u + C.$

830. $du^v = vu^{v-1}.du + \log_e u.u^v.dv.$

If $u = F(y)$ and $y = f(x)$, that is, $u = F(f(x))$, then

831. $\dfrac{du}{dx} = \dfrac{du}{dy}\cdot\dfrac{dy}{dx} = F'(y).f'(x) = D_y u.D_x y.$

Or

832. $du = \dfrac{du}{dy}\cdot\dfrac{dy}{dx}\cdot dx = F'(y).f'(x).dx = D_y u.D_x y.dx.$

833. $\displaystyle\int u.dy = \int u.f'(x)\,dx.$

*Differentials and Integrals of the Simpler Functions of x.*

834. $d\,(a + x) = dx.$ 835. $\displaystyle\int dx = x + C.$

836. $d\,(ax) = adx.$ 837. $\displaystyle\int adx = a\int dx = ax + C.$

838. $d\,(x^n) = nx^{n-1}.dx.$ 839. $\displaystyle\int x^m.dx = \frac{x^{m+1}}{m+1} + C.$

N.B. When $m = -1$, use the next formula.

840. $d\,(\log_e x) = \dfrac{dx}{x}.$ 841. $\displaystyle\int\frac{dx}{x} = \log_e x + C = \log_e cx.$

842. $d\sqrt{x} = \dfrac{dx}{2\sqrt{x}}.$ 843. $\displaystyle\int\frac{dx}{\sqrt{x}} = 2\sqrt{x} + C.$

**844.** $d\,(a + bx)^n = nb\,(a + bx)^{n-1}.$

**845.** $\int (a + bx)^m dx = \frac{(a + bx)^{m+1}}{(m + 1)\,b} + C.$

**846.** $d \log (a + bx) = \frac{bdx}{a + bx}.$

**847.** $\int \frac{dx}{a + bx} = \frac{1}{b} \log_e (a + bx) + C.$

**848.** $d\left(\frac{1}{x}\right) = -\frac{dx}{x^2}.$ **849.** $\int \frac{dx}{x^2} = -\frac{1}{x} + C.$

**850.** $d\left(\frac{1}{a + bx}\right) = \frac{-\,bdx}{(a + bx)^2}.$

**851.** $\int \frac{dx}{(a + bx)^2} = -\frac{1}{b\,(a + bx)} + C.$

**852.** $da^x = \log_e a\,.\,a^x\,.\,dx.$ **853.** $\int a^x.dx = \frac{a^x}{\log_e a} + C.$

**854.** $de^x = e^x\,.\,dx.$ **855.** $\int e^x.dx = e^x + C.$

**856.** $de^{ax} = ae^{ax}.dx.$ **857.** $\int e^{ax}.\,dx = \frac{e^{ax}}{a} + C.$

**858.** $d \log_{10} x = \frac{1}{\log_e 10} \cdot \frac{dx}{x} = \log_{10} e \cdot \frac{dx}{x} = \frac{M\,dx}{x}.$

**859.** $dx^x = x^x (1 + \log_e x)\,dx.$

**860.** $d \sin x = \cos x\,.\,dx.$ **861.** $\int \cos x\,.\,dx = \sin x + C.$

**862.** $d$ covers $x$* $= -\cos x\,.\,dx.$

**863.** $\int \cos x\,.\,dx = -$ covers $x + C.$

**864.** $d \cos x = -\sin x.dx.$ **865.** $\int \sin x\,.\,dx = -\cos x + C.$

**866.** $d$ vers $x$* $= \sin x.dx.$ **867.** $\int \sin x\,.\,dx =$ vers $x + C.$

* The versed sine of $x$ is the difference between 1 and the cosine of $x$. The coversed sine of $x$ is the difference between 1 and the sine of $x$.

vers $x = 1 - \cos x$. covers $x = 1 - \sin x$.

**868.** $d \tan x = \sec^2 x . dx.$ **869.** $\int \sec^2 x . dx = \tan x + C.$

**870.** $d \tan x = \dfrac{2dx}{1 + \cos 2x}.$

**871.** $\displaystyle\int \frac{dx}{1 + \cos x} = \tan \frac{x}{2} + C.$
$= \csc x - \operatorname{ctn} x + C.$

**872.** $\displaystyle\int \frac{dx}{1 + \sin x} = - \tan \left(\frac{\pi}{4} - \frac{x}{2}\right) + C.$

**873.** $d \operatorname{ctn} x = -\csc^2 x . dx.$

**874.** $\int \csc^2 x . dx = - \operatorname{ctn} x + C.$

**875.** $d \operatorname{ctn} x = \dfrac{- 2dx}{1 - \cos 2x}.$

**876.** $\displaystyle\int \frac{dx}{1 - \cos x} = - \operatorname{ctn} \frac{x}{2} + C.$
$= - \operatorname{ctn} x - \csc x + C.$

**877.** $\displaystyle\int \frac{dx}{1 - \sin x} = \tan \left(\frac{\pi}{4} + \frac{x}{2}\right) + C.$

**878.** $d \sec x = \sec x \tan x . dx.$

**879.** $\int \sec x \tan x . dx = \sec x + C.$

**880.** $d \csc x = - \csc x \operatorname{ctn} x . dx.$

**881.** $\int \csc x \operatorname{ctn} x . dx = - \csc x + C.$

**882.** $d \log_e \sin x = \operatorname{ctn} x.$

**883.** $\int \operatorname{ctn} x . dx = \log_e \sin x + C.$

**884.** $d \log_e \cos x = -\tan x.$

**885.** $\int \tan x . dx = - \log_e \cos x + C.$

**886.** $d \log_e \tan x = \sec x \csc x . dx.$

**887.** $\int \sec x \csc x . dx = \log_e \tan x + C.$
$= -\log_e \operatorname{ctn} x + C.$

888. $d \log_e \operatorname{ctn} x = -\sec x \csc x . dx.$

889.

$$\int \sec x . dx = \tfrac{1}{2} \log_e \left(\frac{1+\sin x}{1-\sin x}\right) + C = \log_e \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + C.$$

$$= gd^{-1} x \qquad\qquad = \log_e (\sec x + \tan x) + C.$$

890.

$$\int \csc x . dx = \tfrac{1}{2} \log_e \left(\frac{1-\cos x}{1+\cos x}\right) + C = \log_e \tan \frac{x}{2} + C.$$

$$= -\operatorname{Cosh}^{-1} (\csc x) \qquad\qquad = \log_e (\csc x - \operatorname{ctn} x) + C.$$

891. $d \operatorname{Sinh} x = \operatorname{Cosh} x . dx.$

892. $\int \operatorname{Cosh} x . dx = \operatorname{Sinh} x + C.$

893. $d \operatorname{Cosh} x = \operatorname{Sinh} x . dx.$

894. $\int \operatorname{Sinh} x . dx = \operatorname{Cosh} x + C.$

895. $d \operatorname{Tanh} x = \operatorname{Sech}^2 x . dx.$

896. $\int \operatorname{Sech}^2 x . dx = \operatorname{Tanh} x + C.$

897. $d \operatorname{Ctn} x = -\operatorname{Csch}^2 x . dx.$

898. $\int \operatorname{Csch}^2 x . dx = -\operatorname{Ctn} x + C.$

899. $d \operatorname{Sech} x = -\operatorname{Sech} x . \operatorname{Tanh} x . dx.$

900. $\int \operatorname{Sech} x . \operatorname{Tanh} x . dx = -\operatorname{Sech} x + C.$

901. $d \operatorname{Csch} x = -\operatorname{Csch} x . \operatorname{Ctnh} x . dx.$

902. $\int \operatorname{Csch} x . \operatorname{Ctnh} x . dx = -\operatorname{Csch} x + C.$

903. $d \log_e \operatorname{Sinh} x = \operatorname{Ctnh} x . dx.$

904. $\int \operatorname{Ctnh} x . dx = \log_e \operatorname{Sinh} x + C.$

905. $d \log \operatorname{Cosh} x = \operatorname{Tanh} x . dx.$

906. $\int \operatorname{Tanh} x . dx = \log_e \operatorname{Cosh} x + C.$

907. $d \log_e \operatorname{Tanh} x = \operatorname{Sech} x \operatorname{Csch} x . dx.$

908. $\int \operatorname{Sech} x \operatorname{Csch} x . dx = \log_e \operatorname{Tanh} x + C,$
$= -\log_e \operatorname{Ctnh} x + C.$

909. $d \log_e \operatorname{Ctnh} x = - \operatorname{Sech} x \operatorname{Csch} x . dx.$

910. $\int \operatorname{Sech} x . dx = 2 \tan^{-1} e^x + C.$
$= gdx.$

911. $\int \operatorname{Csch} x . dx = \log \operatorname{Tanh} \frac{x}{2} + C$
$= - \operatorname{Sinh}^{-1} (\operatorname{Csch} x).$

912. $d \sin^{-1} x = \frac{dx}{\sqrt{1 - x^2}}.$

913. $\int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x + C,$
$= - \cos^{-1} x + C.$

914. $d \cos^{-1} x = - \frac{dx}{\sqrt{1 - x^2}}.$

915. $d \tan^{-1} x = \frac{dx}{1 + x^2}.$

916. $\int \frac{dx}{1 + x^2} = \tan^{-1} x + C,$
$= - \operatorname{ctn}^{-1} x + C.$

917. $d \operatorname{ctn}^{-1} x = - \frac{dx}{1 + x^2}.$

918. $d \sec^{-1} x = \frac{dx}{x\sqrt{x^2 - 1}}.$

919. $\int \frac{dx}{x\sqrt{x^2 - 1}} = \sec^{-1} x + C,$
$= - \csc^{-1} x + C.$

920. $d \csc^{-1} x = - \frac{dx}{x\sqrt{x^2 - 1}}.$

921. $d\,\mathrm{vers}^{-1}\,x = \dfrac{dx}{\sqrt{2x - x^2}}.$

922. $$\int \frac{dx}{\sqrt{2x - x^2}} = \mathrm{vers}^{-1}\,x + C,$$
$$= -\,\mathrm{covers}^{-1}\,x + C.$$

923. $d\,\mathrm{covers}^{-1} = -\dfrac{dx}{\sqrt{2x - x^2}}.$

924. $d\,\mathrm{Sinh}^{-1}\,x = \dfrac{dx}{\sqrt{x^2 + 1}}.$

925. $$\int \frac{dx}{\sqrt{x^2 + 1}} = \mathrm{Sinh}^{-1}\,x + C,$$
$$= \log_e\left(x + \sqrt{x^2 + 1}\right) + C,$$
$$= \mathrm{Cosh}^{-1}\sqrt{x^2 + 1}.$$

926. $d\,\mathrm{Cosh}^{-1}\,x = \dfrac{dx}{\sqrt{x^2 - 1}}.$

927. $$\int \frac{dx}{\sqrt{x^2 - 1}} = \mathrm{Cosh}^{-1}\,x + C,$$
$$= \log_e\left(x + \sqrt{x^2 - 1}\right) + C,$$
$$= \mathrm{Sinh}^{-1}\sqrt{x^2 - 1}.$$

928. $d\,\mathrm{Tanh}^{-1}\,x = \dfrac{dx}{1 - x^2}.$

929. $$\int \frac{dx}{1 - x^2} = \mathrm{Tanh}^{-1}\,x + C,$$
$$= \tfrac{1}{2}\log_e\left(\frac{1 + x}{1 - x}\right) + C. \quad x < 1.$$

930. $d\,\mathrm{Ctnh}^{-1}\,x = -\dfrac{dx}{x^2 - 1}.$

931. $$\int \frac{dx}{x^2 - 1} = -\,\mathrm{Ctnh}^{-1}\,x + C,$$
$$= \tfrac{1}{2}\log_e\left(\frac{x - 1}{x + 1}\right) + C. \quad x > 1.$$

932. $d \operatorname{Sech}^{-1} x = -\dfrac{dx}{x\sqrt{1-x^2}}.$

933. $$\int \frac{dx}{x\sqrt{1-x^2}} = -\operatorname{Sech}^{-1} x.$$
$$= -\log_e\left(\frac{1}{x} + \frac{1}{x}\sqrt{1-x^2}\right).$$

934. $d \operatorname{Csch}^{-1} x = -\dfrac{dx}{x\sqrt{1+x^2}}.$

935. $$\int \frac{dx}{x\sqrt{1+x^2}} = -\operatorname{Csch}^{-1} x.$$
$$= -\log_e\left(\frac{1}{x} + \frac{1}{x}\sqrt{1+x^2}\right).$$

*Additional Integrals of Simple Form.*

936. $\int \sin ax . dx = -\dfrac{\cos ax}{a} + C.$

937. $\int \cos ax . dx = \dfrac{\sin ax}{a} + C.$

938. $\int \sin^2 x . dx = -\frac{1}{4}\sin 2x + \frac{1}{2}x + C.$

939. $\int \cos^2 x . dx = \frac{1}{4}\sin 2x + \frac{1}{2}x + C.$

940. $\int \sin x \cos x \, dx = \frac{1}{2}\sin^2 x + C = -\frac{1}{4}\cos 2x + C.$

941. $\int \tan^2 x . dx = \tan x - x + C.$

942. $\int \operatorname{ctn}^2 x . dx = -\operatorname{ctn} x - x + C.$

943. $\int x \sin x . dx = \sin x - x \cos x + C.$

944. $\int x \cos x . dx = \cos x + x \sin x + C.$

945. $\int x \operatorname{Sinh} x \, dx = x \operatorname{Cosh} x - \operatorname{Sinh} x + C.$

946. $\int x \operatorname{Cosh} x \, dx = x \operatorname{Sinh} x - \operatorname{Cosh} x + C.$

947. $\int \operatorname{Sinh}^2 x \, dx = \frac{1}{2} (\operatorname{Sinh} x \operatorname{Cosh} x - x) + C.$

948. $\int \operatorname{Cosh}^2 x \, dx = \frac{1}{2} (\operatorname{Sinh} x \operatorname{Cosh} x + x) + C.$

949. $\int \operatorname{Sinh} x \operatorname{Cosh} x \, dx = \frac{1}{4} \operatorname{Cosh} 2x + C.$

950. $\int \operatorname{Tanh}^2 x \, dx = x - \operatorname{Tanh} x + C.$

951. $\int \operatorname{Ctnh}^2 x \, dx = x - \operatorname{Ctnh} x + C.$

952. $\int e^{ax} \sin nx \, dx = \frac{e^{ax} (a \sin nx - n \cos nx)}{a^2 + n^2} + C.$

953. $\int e^{ax} \cos nx \, dx = \frac{e^{ax} (n \sin nx + a \cos nx)}{a^2 + n^2} + C.$

954. $\int \sin^{-1} x . dx = x \sin^{-1} x + \sqrt{1 - x^2} + C.$

955. $\int \cos^{-1} x . dx = x \cos^{-1} x - \sqrt{1 - x^2} + C.$

956. $\int \tan^{-1} x . dx = x \tan^{-1} x - \frac{1}{2} \log_e (1 + x^2) + C.$

957. $\int \operatorname{ctn}^{-1} x . dx = x \operatorname{ctn}^{-1} x + \frac{1}{2} \log_e (1 + x^2) + C.$

958. $\int \sec^{-1} x \, dx = x \sec^{-1} x - \log_e (x + \sqrt{x^2 + 1}).$

959. $\int \csc^{-1} x \, dx = x \csc^{-1} x + \log_e (x + \sqrt{x^2 + 1}).$

960. $\int \operatorname{vers}^{-1} x \, dx = (x - 1) \operatorname{vers}^{-1} x + \sqrt{2x - x^2}.$

961. $\int \operatorname{Sinh}^{-1} x . dx = x \operatorname{Sinh}^{-1} x - \sqrt{x^2 + 1} + C.$

**962.** $\int \text{Cosh}^{-1} x . dx = x \,\text{Cosh}^{-1} x - \sqrt{x^2 - 1} + C.$

**963.** $\int \text{Tanh}^{-1} x . dx = x \,\text{Tanh}^{-1} x + \frac{1}{2} \log_e (1 - x^2) + C.$

**964.** $\int \text{Ctnh}^{-1} x . dx = x \,\text{Ctnh}^{-1} x + \frac{1}{2} \log_e (x^2 - 1) + C.$

*Successive Differentiation.*

If the derivative $f'(x)$ be differentiated, the result, $f''(x)$, is called the *second* derivative; and if this be differentiated, the result, $f'''(x)$, is called the *third* derivative; and so on, the result of $n$ successive differentiations being denoted by $f^{(n)}(x)$.

The symbols of operation are

$$\frac{d}{dx} \text{ of } \frac{d}{dx} f(x), \text{ or } \frac{d^2}{dx^2} f(x); \; D \text{ of } Df(x), \text{ or } D^2 f(x);$$

$$\frac{d}{dx} \text{ of } \frac{d^2}{dx} f(x), \text{ or } \frac{d^3}{dx^3} f(x); \; D \text{ of } D^2 f(x), \text{ or } D^3 f(x);$$

etc. etc. etc. etc.

**1001.** $$\frac{d^n}{dx^n} f(x) = D^n f(x) = f^{(n)}(x).$$

All three forms of notation are in use.

*Successive Differentiation of a Product of Functions.*

**1002.** $D(uv) = DuD^0v + D^0uDv$, wherein $D^0v = v$ and $D^0u = u$ are written for the sake of symmetry.

**1003.** $D^2(uv) = D^2uD^0v + 2DuDv + D^0uD^2v.$

**1004.** $D^3(uv) = D^3uD^0v + 3D^2uDv + 3DuD^2v + D^0uD^3v.$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

**1005.** $$D^n(uv) = D^nuD^0v + \frac{n}{1} D^{n-1}uDv + \frac{n(n-1)}{1.2} D^{n-2}uD^2v + \ldots + D^0uD^nv.$$

The form of development is that of the Binomial Theorem.

*Taylor's Theorem.*

1006. $$f(x+h) = f(x) + \frac{h}{1}f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots$$

$$\ldots + \frac{h^{n-2}}{(n-2)!}f^{(n-2)}(x) + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(x) + R_n.$$

$$R_n = \frac{h^n}{n!}\, f^{(n)}(x+\theta h), \text{ wherein } 0 < \theta < +1.$$

Or

$$R_n = \frac{(1-\theta)^{n-1}h^n}{(n-1)!}\, f^{(n)}(x+\theta h), \text{ wherein } 0 < \theta < +1.$$

*Maclaurin's Theorem.*

1007. $$f(x) = f(0) + \frac{x}{1}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots$$

$$\ldots + \frac{x^{n-2}}{(n-2)!}f^{(n-2)}(0) + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + R_n.$$

$$R_n = \frac{x^n}{n!}\, f^{(n)}(\theta x), \text{ wherein } 0 < \theta < +1.$$

$$R_n = \frac{(1-\theta)^{n-1}x^n}{(n-1)!}\, f^{(n)}(\theta x), \text{ wherein } 0 < \theta < +1.$$

N.B. — Taylor's and Maclaurin's Theorems are valid only when $f(x)$ and its first $n$ successive derivatives are finite and continuous for all values of $x$ in the interval from $x$ to $x+h$, for Taylor's, and in the interval from 0 to $x$, for Maclaurin's Theorem.

*Taylor's Theorem for a Function of Two Variables.*

$$f(x, y) = u.$$

1008.

$$f(x+h, y+k) = u + (hD_xu + kD_yu)$$
$$+ \frac{1}{2!}(h^2D_x^2u + 2hkD_xD_yu + k^2D_y^2u)$$
$$+ \frac{1}{3!}(h^3D_x^3u + 3h^2kD_x^2D_yu + 3hk^2D_xD_y^2u + k^3D_y^3u) + \ldots$$

the general term being $\frac{1}{n!}(hD_x + kD_y)^nu$.

The symbolic form of Taylor's Theorem for one variable is,

1009. $$f(x+h) = e^{hD_x} f(x),$$

for two variables,

1010. $$f(x+h, y+k) = e^{hD_x + kD_y} f(x, y).$$

*Circular and Hyperbolic Functions expressed in Series.*

1011. $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \dots$ $\quad [x^2 < \infty.$

1012. $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \dots$ $\quad [x^2 < \infty.$

1013. $\tan x = x + \dfrac{x^3}{3} + \dfrac{2x^5}{15} + \dfrac{17x^7}{315} + \dfrac{62x^9}{2835} + \dots$ $\quad [x^2 < \frac{1}{4}\pi^2.$

1014. $\operatorname{ctn} x = \dfrac{1}{x} - \dfrac{x}{3} - \dfrac{x^3}{45} - \dfrac{2x^5}{945} - \dfrac{x^7}{4725} - \dots$ $\quad [x^2 < \pi^2.$

1015. $\sec x = 1 + \dfrac{x^2}{2} + \dfrac{5x^4}{24} + \dfrac{61x^6}{720} + \dots$ $\quad [x^2 < \frac{1}{4}\pi^2.$

1016. $\csc x = \dfrac{1}{x} + \dfrac{x}{6} + \dfrac{7x^3}{360} + \dfrac{31x^5}{15120} + \dots$ $\quad [x^2 < \pi^2.$

1017. $\sin^{-1} x = x + \dfrac{1}{2} \cdot \dfrac{x^3}{3} + \dfrac{1 \times 3}{2 \times 4} \cdot \dfrac{x^5}{5} + \dfrac{1 \times 3 \times 5}{2 \times 4 \times 6} \cdot \dfrac{x^7}{7} + \dots$
$[x^2 < 1.$

1018. $\cos^{-1} x = \frac{1}{2}\pi - \sin^{-1} x.$

1019. $\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots$ $\quad [x^2 < 1.$

1020. $\operatorname{ctn}^{-1} x = \frac{1}{2}\pi - \tan^{-1} x.$

1021. $\tan^{-1} x = \frac{1}{2}\pi - x^{-1} + \frac{1}{3}x^{-3} - \frac{1}{5}x^{-5} + \frac{1}{7}x^{-7} - \dots$
$[x^2 > 1.$

1022. $\sec^{-1} x =$

$\frac{1}{2}\pi - x^{-1} - \dfrac{1}{2} \cdot \dfrac{x^{-3}}{3} - \dfrac{1 \times 3}{2 \times 4} \cdot \dfrac{x^{-5}}{5} - \dfrac{1 \times 3 \times 5}{2 \times 4 \times 6} \cdot \dfrac{x^{-7}}{7} + \dots$ $[x^2 > 1.$

1023. $\csc^{-1} x = \frac{1}{2}\pi - \sec^{-1} x.$

*Bernouillis' and Euler's Numbers.*

Some of the series above given take more symmetrical forms when certain coefficients $B_1$, $B_2$, $B_3$, . . . are used.

**1024.** The coefficients of odd orders, known as Bernouillis' Numbers, are

$$B_1 = \frac{1}{6} \qquad B_{11} = \frac{691}{2730}$$

$$B_3 = \frac{1}{30} \qquad B_{13} = \frac{7}{6}$$

$$B_5 = \frac{1}{42} \qquad B_{15} = \frac{3617}{510}$$

$$B_7 = \frac{1}{30} \qquad B_{17} = \frac{43867}{798}$$

$$B_9 = \frac{5}{66} \qquad \text{etc.} \quad \text{etc.}$$

**1025.** The coefficients of even orders, known as Euler's Numbers, are

$$B_2 = 1 \qquad B_{10} = 50521$$

$$B_4 = 5 \qquad B_{12} = 2702765$$

$$B_6 = 61 \qquad \text{etc.} \quad \text{etc.}$$

$$B_8 = 1385$$

Thus,

**1026.** $\tan x =$

$$\frac{2^2(2^2-1)}{2!}B_1x + \frac{2^4(2^4-1)}{4!}B_3x^3 + \frac{2^6(2^6-1)}{6!}B_5x^5 + \ldots$$

$[x^2 < \frac{1}{4}\pi^2$.

**1027.** $\operatorname{ctn} x = \frac{1}{x} - \frac{2^2}{2!}B_1x - \frac{2^4}{4!}B_3x^3 - \frac{2^6}{6!}B_5x^5 - \frac{2^8}{8!}B_7x^7 - \ldots$

$[x^2 < \pi^2$.

**1028.** $\sec x = 1 + \frac{B_2}{2!}x^2 + \frac{B_4}{4!}x^4 + \frac{B_6}{6!}x^6 + \ldots \qquad [x^2 < \frac{1}{4}\pi^2$.

**1029.** $\csc x =$

$$\frac{1}{x} + \frac{2(2-1)}{2!}B_1x + \frac{2(2^3-1)}{4!}B_3x^3 + \frac{2(2^5-1)}{6!}B_5x^5 + \ldots$$

$[x^2 < \pi^2$.

**1030.** $\operatorname{Sinh} x = \frac{1}{2}(e^x - e^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots$

$[x^2 < \infty$.

**1031.** $\operatorname{Cosh} x = \frac{1}{2}(e^x + e^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots$

$[x^2 < \infty$.

1032. Tanh $x$ =

$$\frac{2^2(2^2-1)}{2!}B_1x-\frac{2^4(2^4-1)}{4!}B_3x^3+\frac{2^6(2^6-1)}{6!}B_5x^5-\ldots$$

$$=x-\frac{x^3}{3}+\frac{2x^5}{15}-\frac{17x^7}{315}+\frac{62x^9}{2835}-\ldots\quad [x^2<\tfrac{1}{4}\pi^2.$$

1033.

$$\text{Ctnh}\,x=\frac{1}{x}+\frac{2^2}{2!}B_1x-\frac{2^4}{4!}B_3x^3+\frac{2^6}{6!}B_5x^5-\frac{2^8}{8!}B_7x^7+\ldots$$

$$=\frac{1}{x}+\frac{x}{3}-\frac{x^3}{45}+\frac{2x^5}{945}-\frac{x^7}{4725}+\ldots\quad [x^2=\pi^2.$$

1034. Sech $x = 1-\frac{B_2}{2!}x^2+\frac{B_4}{4!}x^4-\frac{B_6}{6!}x^6+\frac{B_8}{8!}x^8-\ldots$

$$=1-\frac{x^2}{2}+\frac{5x^4}{24}-\frac{61x^6}{720}+\frac{1385x^8}{40320}-\ldots$$

$$[x^2<\tfrac{1}{4}\pi^2.$$

1035. Csch $x$ =

$$\frac{1}{x}-\frac{2(2-1)}{2!}B_1x+\frac{2(2^3-1)}{4!}B_3x^3-\frac{2(2^5-1)}{6!}B_5x^5+\ldots$$

$$=\frac{1}{x}-\frac{x}{6}+\frac{7x^3}{360}-\frac{31x^5}{15120}+\ldots\quad [x^2<\pi^2.$$

1036. Sinh$^{-1}$ $x = x-\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1\times3}{2\times4}\cdot\frac{x^5}{5}-\frac{1\times3\times5}{2\times4\times6}\cdot\frac{x^7}{7}+\ldots$

$$[x^2<1.$$

1037. Tanh$^{-1}$ $x = x+\frac{1}{3}x^3+\frac{1}{5}x^5+\frac{1}{7}x^7+\ldots\quad [x^2<1.$

$$=\tfrac{1}{2}\log_e\frac{1+x}{1-x}.$$

1038. Ctnh$^{-1}$ $x = x^{-1}+\frac{1}{3}x^{-3}+\frac{1}{5}x^{-5}+\frac{1}{7}x^{-7}+\ldots$

$$=\tfrac{1}{2}\log_e\frac{x+1}{x-1}.\quad [x^2>1.$$

1039. Csch$^{-1}$ $x$ =

$$x^{-1}-\frac{1}{2}\cdot\frac{x^{-3}}{3}+\frac{1\times3}{2\times4}\cdot\frac{x^{-5}}{5}-\frac{1\times3\times5}{2\times4\times6}\cdot\frac{x^{-7}}{7}+\ldots[x^2>1.$$

1040. $\dfrac{x}{e^x-1}=1-\dfrac{x}{2}+\dfrac{B_1x^2}{2!}-\dfrac{B_3x^4}{4!}+\dfrac{B_5x^6}{6!}-\dots\quad [x<2\pi.$

1041. $\dfrac{x}{e^x+1}=$

$$\frac{x}{2}-\frac{2^2-1}{2!}B_1x^2+\frac{2^4-1}{4!}B_3x^4-\frac{2^6-1}{6!}B_5x^6+\dots$$

1042. $\dfrac{1}{2}\cdot\dfrac{e^x-1}{e^x+1}=\dfrac{2^2-1}{2!}B_1x-\dfrac{2^4-1}{4!}B_3x^3+\dfrac{2^6-1}{6!}B_5x^5-\dots$

*Evaluation of Indeterminate Forms.*

1051. $\dfrac{0}{0}$. When the fraction $\dfrac{f(x)}{F(x)}$ for any given value of the variable, $x=a$, takes the form $\dfrac{0}{0}$, its true value can be found by differentiating both numerator and denominator and in the result, $\dfrac{f'(x)}{F'(x)}$, substituting $a$ for $x$. If $\dfrac{f'(x)}{F'(x)}$ also takes the form $\dfrac{0}{0}$, differentiate again, and, if necessary, again and again until a result is found which does not take the form $\dfrac{0}{0}$ for $x=a$. This last result will give the true value sought.

1052. $\dfrac{\infty}{\infty}$. When the fraction assumes this form, the true value is found in the same way as in the case of $\dfrac{0}{0}$.

1053. $0.\infty$. When, for $x=a$, $f(x)=0$ and $\varphi(x)=\infty$, so that the product $f(x).\varphi(x)$ takes the form $0.\infty$, the true value of the product may be found by putting $\varphi(x)=\dfrac{1}{F(x)}$ and finding that of $\dfrac{f(x)}{F(x)}$, which takes the form $\dfrac{0}{0}$.

1054. $0^0$, $1^\infty$, $\infty^0$. When, for $x=a$, the expression $\varphi(x)^{f(x)}$ takes either of these forms, its logarithm may be taken and dealt with as in the last case. Thus, putting

$$\varphi(x)^{f(x)}=y,\qquad \log_e y=f(x).\log_e(\varphi(x)),$$
$$y=e^{f(x).\log_e(\phi(x))}.$$

The true value of the exponent, $f(x) . \log_e (\varphi(x))$, having been found, that of $y$, or that of the given function, follows.

1055. $\infty - \infty$. When $f(x) - F(x)$ takes the form $\infty - \infty$ it can be changed to a form which becomes $\frac{0}{0}$, and dealt with as in the first case.

Thus,

$$f(x) - F(x) = \frac{\frac{1}{F(x)} - \frac{1}{f(x)}}{\frac{1}{f(x).F(x)}} = \frac{\frac{1}{\infty} - \frac{1}{\infty}}{\frac{1}{\infty . \infty}} = \frac{0-0}{0} = \frac{0}{0}.$$

*Partial Differential Coefficients.*

A differential coefficient of a function of two or more variables, $f(x, y, z \ldots)$, obtained on the supposition that all the variables save one are, for the time being, constants, is a *partial differential coefficient.*

It is written

when $x$ alone is supposed to vary, $\frac{\partial}{\partial x} f(x, y, z \ldots)$,

when $y$ alone is supposed to vary, $\frac{\partial}{\partial y} f(x, y, z \ldots)$,

when $z$ alone is supposed to vary, $\frac{\partial}{\partial z} f(x, y, z \ldots)$,

the form of the letter $\partial$ indicating that the differentiation is partial.

Partial differential coefficients are also written

$D_x f(x, y, z \ldots) \quad D_y f(x, y, z \ldots) \quad D_z f(x, y, z \ldots)$ etc.

the subscript letters $x$, $y$, $z$, etc., indicating the variables with reference to which the partial differentiation takes place.

Often the single letter $f$ is used as an abbreviation for $f(x, y, z \ldots)$, and $df$ for the total differential of the same. Thus

1056. $$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz + \ldots$$

or

1057. $$df = D_x f . dx + D_y f . dy + D_z f . dz + \ldots$$

1058. $D_z D_y D_x f = D_x D_z D_y f = D_y D_x D_z f$, means that the order in which the several partial differentiations $D_x$, $D_y$, $D_z$, are performed is of no consequence. The commutative law.

If $u = f(x, y)$,

1059. $$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \text{ or } du = D_x u . dx + D_y u . dy.$$

1060. $$d^2 u = D_x{}^2 u . dx^2 + 2 D_x D_y u . dx\, dy + D_y{}^2 u . dy^2.$$

1061. $$d^3 u = D_x{}^3 u . dx^3 + 3 D_x{}^2 D_y u . dx^2\, dy + 3 D_x D_y{}^2 u . dx\, dy^2 + D_y{}^3 u . dy^3.$$

1062. Or, symbolically, $d^3 u = (D_x . dx + D_y . dy)^3\, u$.

1063. And, in general, $d^n u = (D_x . dx + D_y . dy)^n u$.

1064. $$df(x, y, z \ldots) = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz + \ldots$$
$$= D_x f . dx + D_y f . dy + D_z f . dz + \ldots$$

1065. $$d^n f(x, y, z \ldots) = (D_x . dx + D_y . dy + D_z dz + \ldots)^n f.$$

1066. If $u = F(x, y)$ and $y = f(x)$, $\dfrac{du}{dx} = D_x u + D_y u . \dfrac{dy}{dx}$.

1067. If $u = F(y, z)$ and $y = f_1(x)$
$z = f_2(x)$,

then
$$\frac{du}{dx} = D_y u . \frac{dy}{dx} + D_z u\, \frac{dz}{dx}.$$

1068. $$\frac{d^2 u}{dx^2} = D_y{}^2 u \left(\frac{dy}{dx}\right)^2 + 2 D_y D_z u . \frac{dy}{dx}\frac{dz}{dx} + D_z{}^2 u \left(\frac{dz}{dx}\right)^2 + D_y u . \frac{d^2 y}{dx^2} + D_z u\, \frac{d^2 z}{dx^2}.$$

NOTE: $\dfrac{dy}{dx}$ and $\dfrac{dz}{dx}$, being explicit functions of $x$, are treated as constants in finding the partial derivatives relatively to $y$ and $z$.

If $x$ and $y$ be functions of an independent variable $t$, then

1069. $$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}, \quad \text{or} \quad D_x y = \frac{D_t y}{D_t x}.$$

1070. $$\frac{d^2y}{dx^2}=\frac{\frac{dx}{dt}\cdot\frac{d^2y}{dt^2}-\frac{dy}{dt}\cdot\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3},$$

or $$D_x^2y=\frac{D_tx\,.\,D_t^2y-D_ty\,.\,D_t^2x}{(D_tx)^3}.$$

1071.

$$\frac{d^3y}{dx^3}=\frac{\left(\frac{dx}{dt}\right)^2\cdot\frac{d^3y}{dt^3}-3\frac{dx}{dt}\cdot\frac{d^2x}{dt^2}\cdot\frac{d^2y}{dt^2}+3\frac{dy}{dt}\cdot\left(\frac{d^2x}{dt^2}\right)^2-\frac{dx}{dt}\cdot\frac{dy}{dt}\cdot\frac{d^3x}{dt^3}}{\left(\frac{dx}{dt}\right)^5}.$$

or

$$D_x^3y=\frac{(D_tx)^2D_t^3y-3D_tx\,.\,D_t^2x\,.\,D_t^2y+3D_ty\,(D_t^2x)^2-D_tx\,.\,D_ty\,.\,D_t^3x}{(D_tx)^5}.$$

*Change of Independent Variable.*

In formulas **1069**, **1070** and **1071**, put $t=y$. The result is the following three formulas for changing the independent variable from $x$ to $y$, that is when $y=f(x)$ becomes $x=\phi(y)$.

1072. $$\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}},\quad\text{or } D_xy=\frac{1}{D_yx}.$$

1073. $$\frac{d^2y}{dx^2}=-\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^3},\quad\text{or } D_x^2y=-\frac{D_y^2x}{(D_yx)^3},$$

1074. $$\frac{d^3y}{dx^3}=\frac{3\left(\frac{d^2x}{dy^2}\right)^2-\frac{dx}{dy}\cdot\frac{d^3x}{dy^3}}{\left(\frac{dx}{dy}\right)^5},$$

or $$D_x^3y=\frac{3\,(D_y^2x)^2-D_yx\,.\,D_y^3x}{(D_yx)^5}.$$

To find $D_x y$, $D_x^2 y$, etc., when $y$ is an implicit function of $x$, $f(x, y) = 0$.

1075.
$$\begin{cases} \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y} \cdot \dfrac{dy}{dx} = 0, \text{ whence } \dfrac{dy}{dx} = -\dfrac{\partial f}{\partial x} \div \dfrac{\partial f}{\partial y} \\ \text{Or} \\ D_x f + D_y f \,.\, D_x y = 0, \text{ whence } D_x y = -D_x f \div D_y f. \end{cases}$$

1076. $D_x^2 f + 2D_x D_y f \,.\, (D_x y) + D_y^2 f \,.\, (D_x y)^2 + D_y f \,.\, D_x^2 y = 0$, which, by substituting the value of $D_x y$ found in 1075, gives $D_x^2 y$.

1077.
$$\begin{aligned} D_x^3 f &+ 3D_x^2 D_y f \,.\, (D_x y) + 3D_x D_y^2 f \,.\, (D_x y)^2 \\ &+ D_y^3 f \,.\, (D_x y)^3 + 3D_x D_y f \,.\, (D_x^2 y) \\ &+ 3D_y^2 f \,.\, (D_x y)(D_x^2 y) + D_y f \,.\, D_x^3 y = 0, \end{aligned}$$

which, by substituting the values of $D_x y$ and $D_x^2 y$ found in 1075 and 1076, gives $D_x^3 y^2$.

### *Maxima and Minima.*

1078. *Functions of one variable.* The values of $x$ which give a maximum or a minimum value to $y = f(x)$ are found by solving the equation $f'(x) = 0$. Any root, $a$, being substituted for $x$,

if $\quad f''(a) < 0, \quad f(a)$ is a maximum,

if $\quad f''(a) > 0, \quad f(a)$ is a minimum.

If $f''(a) = 0$ and $f'''(a)$ is not 0, $f(a)$ is neither a maximum nor a minimum; but if $f'''(a)$ is also 0, $f(a)$ is a maximum or a minimum according as $f''''(a) \lessgtr 0$.

And in general, if the first derivative which does not vanish for $x = a$ is of an *odd* order $f(a)$ is neither a maximum nor a minimum; but if it is of an even order, the $2n^{th}$, then $f(a)$ is a maximum or a minimum according as $f^{(2n)}(a) \lessgtr 0$.

**1079.** *Implicit functions.* $f(x, y) = 0$. The values of $x$ which give a maximum or a minimum value to $y$ must satisfy the equations

$$\frac{\partial f(x,y)}{\partial x} = 0 \text{ and } f(x, y) = 0,$$

as well as fulfill the condition $\dfrac{\partial f(x, y)}{\partial y} \gtrless 0.$

$y$ is a maximum when $\left[-\dfrac{\partial^2 f}{\partial x^2}\right] \div \left[\dfrac{\partial f}{\partial y}\right] < 0,$

$y$ is a minimum when $\left[-\dfrac{\partial^2 f}{\partial x^2}\right] \div \left[\dfrac{\partial f}{\partial y}\right] > 0.$

**1080.** *Function of two independent variables.*

$u = f(x, y)$ will be a maximum or a minimum for values of $x$ and $y$ that satisfy the two equations

$$\frac{\partial}{\partial x} f(x, y) = 0, \quad \frac{\partial}{\partial y} f(x, y) = 0,$$

and also the condition

$$\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 > 0.$$

$\left.\begin{array}{l}\text{A maximum}\\ \text{A minimum}\end{array}\right\}$ requires $\dfrac{\partial^2 f}{\partial x^2}$ and $\dfrac{\partial^2 f}{\partial y^2}$ both to be $\left\{\begin{array}{l}\text{negative}\\ \text{positive.}\end{array}\right.$

*Integration of Rational Algebraic Functions.*

**1081.** Rational algebraic functions are either of the form

$$Ax^m + Bx^{m-1} + Cx^{m-2} + \ldots + Kx + L,$$

or of the form

$$\frac{Ax^m + Bx^{m-1} + Cx^{m-2} + \ldots + Kx + L}{x^n + ax^{n-1} + bx^{n-2} + \ldots + kx + l}.$$

In the former case the function is integrated by integrating each term separately. Formulas **816** and **839**. In the latter case, if the degree $m$ of the numerator is not less

than the degree $n$ of the denominator, the numerator is divided by the denominator until a remainder appears which is of a degree at least one less than $n$. Thus the function to be integrated takes the form

$$A_0x^d + A_1x^{d-1} + A_2x^{d-2} + \ldots + A_d + \frac{f(x)}{F(x)}.$$

where $d = m - n$, and the degree of $f(x)$ is not higher than $n - 1$.

The first part of this expression is integrated, as before, by integrating its separate terms; and there remains to be integrated the fraction $\frac{f(x)}{F(x)}$, the degree of the numerator being less than that of the denominator. Such a fraction is called a *rational proper fraction.*

*Integration of Rational Proper Fractions.*

A rational proper fraction is of the general form

1082. $$\frac{f(x)}{F(x)} = \frac{Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \ldots + Kx + L}{x^n + ax^{n-1} + bx^{n-2} + \ldots + kx + l}.$$

This is to be separated into partial fractions the sum of which shall be equal to $\frac{f(x)}{F(x)}$. For this purpose $F(x)$ is separated into factors by solving the equation $F(x) = 0$ to find the roots $x_1, x_2, x_3, \ldots x_n$, and then putting

$$F(x) = (x - x_1)(x - x_2)(x - x_3) \ldots (x - x_n).$$

1083. Now assume

$$\frac{f(x)}{F(x)} = \frac{A_1}{x - x_1} + \frac{A_2}{x - x_2} + \ldots + \frac{A_n}{x - x_n}$$

where the constants $A_1, A_2, \ldots A_n$ are to be determined. This is done by clearing the equation of fractions, which gives the identity

$$f(x) = A_1f_1(x) + A_2f_2(x) + \ldots + A_nf_n(x),$$

and equating the coefficients of like powers of $x$ on each side of this identity.

Thus the function to be integrated becomes the sum of $n$ partial fractions each of the form $\frac{A}{x-a}$, which is integrable by **847**.

$$\int \frac{A dx}{x-a} = A \log_e (x-a) + C.$$

Thus far it has been assumed that the $n$ roots of the equation $F(x) = 0$ are all real and different.

**1084.** If $F(x) = 0$ have equal roots, say $p$ roots each equal to $x_1$, $q$ roots each equal to $x_2$, and so on, then

$$F(x) = (x - x_1)^p (x - x_2)^q (x - x_3)^r \ldots$$

where $$p + q + r + \ldots = n.$$

The $n$ assumed partial fractions are

$$\begin{aligned} \frac{f(x)}{F(x)} = & \frac{A_1}{(x-x_1)^p} + \frac{A_2}{(x-x_1)^{p-1}} + \ldots + \frac{A_p}{x-x_1} \\ & + \frac{B_1}{(x-x_2)^q} + \frac{B_2}{(x-x^2)^{q-1}} + \ldots + \frac{B_q}{x-x_2} \\ & + \ \ldots . \qquad \ldots . \qquad \ldots . \end{aligned}$$

The values of the numerators are found, as before, by clearing of fractions and equating the coefficients of like powers of $x$.

Thus the function to be integrated becomes the sum of $n$ partial fractions, some of the form $\frac{A}{x-a}$, already integrated, and the rest of the form $\frac{A}{(x-a)^k}$, which is integrable by **845**.

$$\int \frac{A}{(x-a)^k} = \frac{-A}{(k-1)(x-a)^{k-1}} + C.$$

**1085.** If $F(x) = 0$ have imaginary roots, these come in pairs; that is, if $x_1$ is of the form $p + iq$ some other root, $x_2$ say, is of the conjugate form $p - iq$. The corresponding partial fractions being added give a fraction of the form $\frac{Px + Q}{(x-p)^2 + q^2}$, which is free from imaginaries and is used in place of the two partial fractions that were added to form it.

If there be two or more, say $l$, equal imaginary roots and their conjugates, the corresponding partial fractions to be assumed are

$$\frac{P_1x+Q_1}{[(x-p)^2+q^2]^l}+\frac{P_2x+Q_2}{[(x-p)^2+q^2]^{l-1}}+\ldots+\frac{P_lx+Q_l}{(x-p)^2+q^2}.$$

And similarly for any other set of equal imaginary roots. The values of the $P$'s and $Q$'s, as well as of any other assumed constants in the numerators of other partial fractions, are found, as before, by clearing the equation of fractions and equating coefficients of like powers of $x$. Thus the function to be integrated is broken up into partial fractions, some, it may be, of forms already integrated and the others of the forms $\dfrac{Px+Q}{(x-p)^2+q^2}$ and $\dfrac{Px+Q}{[(x-p)^2+q^2]^l}$, which are integrable by the next ten formulas.

**1086.**

$$\int\frac{A+Bx}{a+bx+cx^2}dx=A\int\frac{dx}{a+bx+cx^2}+B\int\frac{x\,dx}{a+bx+cx^2}.$$

**1087.** $$\int\frac{dx}{a+bx+cx^2}=\frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\frac{b+2cx}{\sqrt{4ac-b^2}}+C,$$
when $4ac-b^2>0$.

**1088.** $$\int\frac{dx}{a+bx+cx^2}=\frac{-2}{b+2cx}+C,$$ when $4ac-b^2=0$.

**1089.** $$\int\frac{dx}{a+bx+cx^2}=$$
$$\frac{1}{\sqrt{b^2-4ac}}\log_e\left(\frac{\sqrt{(b^2-4ac)}-b-2cx}{\sqrt{(b^2-4ac)}+b+2cx}\right)+C,$$
when $4ac-b^2<0$.

**1090.** $$\int\frac{dx}{a+bx+cx^2}=$$
$$\frac{1}{\sqrt{b^2-4ac}}\log_e\left(\frac{b+2cx-\sqrt{b^2-4ac}}{b+2cx+\sqrt{b^2-4ac}}\right)+C,$$

which is merely another form of **1089**, to be used when $\sqrt{b^2-4ac}$ is less than $b+2cx$.

**1091.** $$\int \frac{dx}{a + bx + cx^2} = \frac{-2}{\sqrt{b^2 - 4ac}} \operatorname{Tanh}^{-1} \frac{b + 2cx}{\sqrt{b^2 - 4ac}} + C,$$

which is analogous to **1087**, and to be used when $4ac - b^2 < 0$, and $b + 2cx < \sqrt{b^2 - 4ac}$. If $b + 2cx > \sqrt{b^2 - 4ac}$, $\operatorname{Ctnh}^{-1}$ may be written in place of $\operatorname{Tanh}^{-1}$.

**1092.** $$\int \frac{x\,dx}{a + bx + cx^2} = \frac{1}{2c} \log_e (a + bx + cx^2) - \frac{b}{2c} \int \frac{dx}{a + bx + cx^2}.$$

**1093.** $$\int \frac{x\,dx}{(a + bx + cx^2)^p} = \frac{-1}{2c\,(p - 1)\,(a + bx + cx^2)^{p-1}} - \frac{b}{2c} \int \frac{dx}{(a + bx + cx^2)^p}.$$

**1094.** $$\int \frac{dx}{(a + bx + cx^2)^p} = \frac{1}{4ac - b^2} \cdot \frac{b + 2cx}{(p - 1)\,(a + bx + cx^2)^{p-1}} + \frac{(2p - 3)2c}{(4ac - b^2)\,(p - 1)} \int \frac{dx}{(a + bx + cx^2)^{p-1}}.$$

The repeated application of this reduction formula makes the integral in the first member depend ultimately on

$$\int \frac{dx}{a + bx + cx^2},$$ for which refer above to **1087–1091**.

Formula **1086** by substitution from **1092** becomes

**1095.** $$\int \frac{A + Bx}{a + bx + cx^2} = \frac{B}{2c} \log_e (a + bx + cx^2) + \frac{2Ac - Bb}{2c} \int \frac{dx}{a + bx + cx^2}.$$

This completes the list of formulas necessary for the integration of rational proper fractions by breaking them into partial fractions.

A few formulas easily deduced from the foregoing are added here.

1096. $\int \frac{dx}{a + cx^2} = \frac{1}{\sqrt{ac}} \tan^{-1} \frac{cx}{\sqrt{ac}} + C.$ $[a > 0,\ c > 0.]$

1097. $\int \frac{dx}{cx^2} = -\frac{1}{cx} + C.$

1098. $\left.\begin{aligned}\int \frac{dx}{a - cx^2} &= \frac{1}{\sqrt{ac}} \operatorname{Tanh}^{-1} \frac{cx}{\sqrt{ac}} + C.\\ &= \frac{1}{2\sqrt{ac}} \log_e \left(\frac{\sqrt{ac} + cx}{\sqrt{ac} - cx}\right) + C.\end{aligned}\right\} \begin{matrix} a > 0, \\ c > 0.\end{matrix}$

1099. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + C = -\frac{1}{a} \operatorname{ctn}^{-1} \frac{x}{a} + C.$

1100. $\int \frac{dx}{a^2 - x^2} = \frac{1}{a} \operatorname{Tanh}^{-1} \frac{x}{a} + C.$ $[x < a]$

$= \frac{1}{a} \log_e \sqrt{\frac{a + x}{a - x}} + C.$ $[x < a]$

1101. $\int \frac{dx}{x^2 - a^2} = -\frac{1}{a} \operatorname{Ctnh}^{-1} \frac{x}{a} + C$

$= \frac{1}{a} \log_e \sqrt{\frac{x - a}{x + a}} + C.$ $[x > a]$

*Integration of Irrational Algebraic Functions.*

An expression containing radicals can very often be changed to a rational form by the substitution of a new variable. Following are cases in which this device is useful.

1102. Put $\sqrt{a + bx} = y$, then $x = \frac{y^2 - a}{b}$, $dx = \frac{2y}{b} dy.$

1103. Put $\sqrt[n]{a + bx} = y$, then $x = \frac{y^n - a}{b}$, $dx = \frac{ny^{n-1}}{b} dy.$

**1104.** For the removal of a radical of the form

$$\sqrt{a^2+b^2x^2}, \text{ put } \frac{\sqrt{a^2+b^2x^2}-bx}{a}=y, \text{ then}$$

$$x=\frac{a}{b}\cdot\frac{1-y^2}{2y},$$

$$\sqrt{a^2+b^2x^2}=\frac{a\,(1+y^2)}{2y},$$

$$dx=-\frac{a}{b}\cdot\frac{1+y^2}{2y^2}\,dy.$$

**1105.** For the removal of a radical of the form $\sqrt{a^2-b^2x^2}$, put

$$\sqrt{\frac{a-bx}{a+bx}}=y, \text{ then } x=\frac{a}{b}\cdot\frac{1-y^2}{1+y^2},$$

$$\sqrt{a^2-b^2x^2}=\frac{2ay}{1+y^2},$$

$$dx=-\frac{4a}{b}\cdot\frac{y\,dy}{(1+y^2)^2}.$$

For the removal of a radical of the form $\sqrt{a+bx\pm cx^2}$, where $c$ in itself is always positive, there are two devices, one for the upper sign and one for the lower.

**1106.** (i) Put, for abbreviation, $\sqrt{4ac-b^2}=h$, and then put

$$\frac{2\sqrt{c}\sqrt{a+bx+cx^2}-(b+2cx)}{h}=y, \text{ which gives}$$

$$x=\frac{h\,(1-y^2)-2by}{4cy},$$

$$\sqrt{a+bx+cx^2}=\frac{h\,(1+y^2)}{4\sqrt{c}\,.\,y},$$

$$dx=-\frac{h\,(1+y^2)}{4cy^2}\,dy.$$

**1107.** (ii) Put, for abbreviation, $\sqrt{4ac+b^2}=k$, and then put

$$\sqrt{\frac{k+b-2cx}{k-b+2cx}}=y, \text{ which gives}$$

$$x=\frac{k+b-(k-b)\,y^2}{2c\,(1+y^2)}$$

$$\sqrt{a+bx-cx^2}=\frac{ky}{\sqrt{c}.(1+y^2)}$$

$$dx=-\frac{2kydy}{c\,(1+y^2)^2}.$$

**1108.** $$\int\frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\frac{x}{a}+C, \text{ or }=-\cos^{-1}\frac{x}{a}+C.$$

**1109.** $$\int\frac{x^2\,dx}{\sqrt{a^2-x^2}}=-\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C.$$

**1110.** $$\int\sqrt{a^2-x^2}.dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C.$$

**1111.** $$\int\frac{dx}{\sqrt{x^2+a^2}}=\text{Sinh}^{-1}\frac{x}{a}+C.$$
$$=\log_e(x+\sqrt{x^2+a^2})+C.$$

**1112.** $$\int\frac{x^2\,dx}{\sqrt{x^2+a^2}}=\frac{x}{2}\sqrt{x^2+a^2}-\frac{a^2}{2}\text{Sinh}^{-1}\frac{x}{a}+C.$$
$$=\frac{x}{2}\sqrt{x^2+a^2}-\frac{a^2}{2}\log_e(x+\sqrt{x^2+a^2})+C.$$

**1113.** $$\int\sqrt{x^2+a^2}.dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\text{Sinh}^{-1}\frac{x}{a}+C$$
$$=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log_e(x+\sqrt{x^2+a^2})+C.$$

**1114.** $$\int\frac{dx}{\sqrt{x^2-a^2}}=\text{Cosh}^{-1}\frac{x}{a}+C.$$
$$=\log_e(x+\sqrt{x^2-a^2})+C.$$

1115. $$\int \frac{x^2 dx}{\sqrt{x^2 - a^2}} = \frac{x}{2}\sqrt{x^2 - a^2} + \frac{a^2}{2}\operatorname{Cosh}^{-1}\frac{x}{a} + C.$$

$$= \frac{x}{2}\sqrt{x^2 - a^2} + \frac{a^2}{2}\log_e(x + \sqrt{x^2 - a^2}) + C.$$

1116. $$\int \sqrt{x^2 - a^2}\,.\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\operatorname{Cosh}^{-1}\frac{x}{a}$$

$$= \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log_e(x + \sqrt{x^2 - a^2}) + C.$$

1117. $$\int \frac{dx}{\sqrt{2ax - x^2}} = \operatorname{vers}^{-1}\frac{x}{a} + C = \sin^{-1}\frac{x - a}{a} + C.$$

1118. $$\int \frac{x\,dx}{\sqrt{2ax - x^2}} = -\sqrt{2ax - x^2} + a\operatorname{vers}^{-1}\frac{x}{a} + C$$

$$= -\sqrt{2ax - x^2} + a\sin^{-1}\frac{x - a}{a} + C.$$

1119. $$\int \sqrt{2ax - x^2}.dx =$$

$$\frac{x - a}{2}\sqrt{2ax - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x - a}{a}\right) + C.$$

1120.

$$\int \frac{dx}{\sqrt{a + bx + cx^2}} = \frac{1}{\sqrt{c}}\log_e\left(\frac{b + 2cx}{2} + \sqrt{c}\sqrt{a + bx + cx^2}\right) + C,$$

when $c > 0$.

$$= \frac{1}{\sqrt{c}}\operatorname{Sinh}^{-1}\left(\frac{b + 2cx}{\sqrt{4ac - b^2}}\right) + C,$$

when $4ac - b^2 > 0,\ c > 0$.

$$= \frac{1}{\sqrt{c}}\operatorname{Cosh}^{-1}\left(\frac{b + 2cx}{\sqrt{b^2 - 4ac}}\right) + C,$$

when $4ac - b^2 < 0,\ c > 0$.

$$= \frac{-1}{\sqrt{-c}}\sin^{-1}\left(\frac{b + 2cx}{\sqrt{b^2 - 4ac}}\right) + C, \text{ when } c < 0.$$

1121.

$$\int \sqrt{a + bx + cx^2}\,.\,dx = \frac{b + 2cx}{4c}\sqrt{X} + \frac{4ac - b^2}{8c}\int \frac{dx}{\sqrt{X}},$$

wherein $\sqrt{X} = \sqrt{a + bx + cx^2}$.

1122.

$$\int \frac{(A+Bx)\,dx}{\sqrt{a+bx+cx^2}} = \frac{B\sqrt{X}}{c} + \frac{2Ac-Bb}{2c}\int \frac{dx}{\sqrt{X}},$$

wherein $\sqrt{X} = \sqrt{a+bx+cx^2}$.

1123.

$$\int \frac{x^m\,dx}{\sqrt{a+bx+cx^2}} = \frac{x^{m-1}\sqrt{X}}{mc} - \frac{(2m-1)b}{2mc}\int \frac{x^{m-1}\,dx}{\sqrt{X}}$$

$$- \frac{(m-1)\,a}{mc}\int \frac{x^{m-2}\,dx}{\sqrt{X}},$$

wherein $\sqrt{X} = \sqrt{a+bx+cx^2}$.

1124. $\int \frac{dx}{(x-p)^m\sqrt{a+bx+cx^2}}$, by the substitution of $y$ for $\frac{1}{x-p}$, is changed to the form of the first member of the preceding formula.

1125.

$$\left.\begin{aligned} \int \frac{dx}{(a-x)\sqrt{x-b}} &= \frac{2}{\sqrt{a-b}}\operatorname{Tanh}^{-1}\sqrt{\frac{x-b}{a-b}} + C \\ \text{or} \quad &= \frac{2}{\sqrt{a-b}}\operatorname{Ctnh}^{-1}\sqrt{\frac{x-b}{a-b}} + C \\ \text{or} \quad &= \frac{-2}{\sqrt{b-a}}\tan^{-1}\sqrt{\frac{x-b}{b-a}} + C \end{aligned}\right\} \text{The real form to be taken.}$$

1126.

$$\left.\begin{aligned} \int \frac{dx}{(a-x)\sqrt{b-x}} &= \frac{2}{\sqrt{b-a}}\operatorname{Tanh}^{-1}\sqrt{\frac{b-x}{b-a}} + C \\ \text{or} \quad &= \frac{2}{\sqrt{b-a}}\operatorname{Ctnh}^{-1}\sqrt{\frac{b-x}{b-a}} + C \\ \text{or} \quad &= \frac{-2}{\sqrt{a-b}}\tan^{-1}\sqrt{\frac{b-x}{a-b}} + C \end{aligned}\right\} \text{The real form to be taken.}$$

1127. $\int \frac{(A + Bx)\, dx}{(f + gx + hx^2)\sqrt{a + bx + cx^2}}$, by putting $\frac{py + q}{y + 1}$

in place of $x$, the values of $p$ and $q$ being determined from the equations

$$2cpq + b\,(p + q) + 2a = 0$$
$$2hpq + g\,(p + q) + 2f = 0$$

and then putting $z$ in place of $y^2$, is reduced to the forms

$\int \frac{dz}{(f_1 + h_1 z)\sqrt{a_1 + c_1 z}}$ and $\int \frac{dz}{(f_2 + h_2 z)\sqrt{a_1 z + c_1 z^2}}$, which are

integrable by 1102, 1096, and 1098.

1128. $\int \frac{f\,(x)\, dx}{F\,(x)\sqrt{a + bx + cx^2}}$ can be integrated by the separation of $\frac{f\,(x)}{F\,(x)}$ into a series of terms and partial fractions, multiplying each by $\frac{dx}{\sqrt{a + bx + cx^2}}$, and integrating the separate terms. The partial fractions will be of the form 1124 or, when a pair of imaginary roots of the equation $F\,(x) = 0$ occurs, of the form 1127.

1129. $$\int \frac{(A + Bx)\, dx}{(a + bx + cx^2)^{\frac{3}{2}}} = \frac{(4aB - 2bA) + (2bB - 4cA)x}{(b^2 - 4ac)\sqrt{a + bx + cx^2}} + C.$$

*Reduction formulas for integral powers of the trigonometric functions.*

1130. $$\int \sin^m x \,.\, \cos^n x \,.\, dx$$
$$= -\frac{\sin^{m-1} x \,.\, \cos^{n+1} x}{m + n} + \frac{m - 1}{m + n}\int \sin^{m-2} x \,.\, \cos^n x \,.\, dx,$$

1131. $$= \frac{\sin^{m+1} x \cos^{n-1} x}{m + n} + \frac{n - 1}{m + n}\int \sin^m x \,.\, \cos^{n-2} x \,.\, dx.$$

1132. $$\int \frac{\sin^m x}{\cos^n x}\, dx$$
$$= \frac{\sin^{m+1} x}{(n - 1)\cos^{n-1} x} + \frac{n - m - 2}{n - 1}\int \frac{\sin^m x}{\cos^{n-2} x}\, dx.$$

1133. $$= -\frac{\sin^{m-1} x}{(m-n)\cos^{n-1} x} + \frac{m-1}{m-n}\int \frac{\sin^{m-2} x}{\cos^n x}\,dx.$$

1134. $$= \frac{\sin^{m-1} x}{(n-1)\cos^{n-1} x} - \frac{m-1}{n-1}\int \frac{\sin^{m-2} x}{\cos^{n-2} x}\,dx.$$

1135. $$\int \frac{\cos^n x}{\sin^m x}\,dx$$
$$= -\frac{\cos^{n+1} x}{(m-1)\sin^{m-1} x} + \frac{m-n-2}{m-1}\int \frac{\cos^n x}{\sin^{m-2} x}\,dx.$$

1136. $$= -\frac{\cos^{n-1} x}{(m-1)\sin^{m-1} x} - \frac{n-1}{m-1}\int \frac{\cos^{n-2} x}{\sin^{m-2} x}\,dx.$$

1137. $$\int \frac{dx}{\sin^m x \cos^n x}$$
$$= \frac{1}{(n-1)\sin^{m-1} x.\cos^{n-1} x} + \frac{m+n-2}{n-1}\int \frac{dx}{\sin^m x.\cos^{n-2} x}.$$

1138.
$$= \frac{-1}{(m-1)\sin^{m-1} x \cos^{n-1} x} + \frac{m+n-2}{m-1}\int \frac{dx}{\sin^{m-2} x \cos^n x}.$$

The result of repeated applications of one or of two (alternately) of the foregoing formulas (1130–1138) is one of the following integrable forms:

$$\int dx = x,$$

$$\int \cos x.dx = \sin x, \qquad \int \frac{\cos x}{\sin x}\,dx = \log_e \sin x,$$

$$\int \sin x.dx = -\cos x, \qquad \int \sin x \cos x.dx = -\tfrac{1}{4}\cos 2x,$$

$$\int \frac{\sin x}{\cos x}\,dx = -\log_e \cos x, \qquad \int \frac{dx}{\sin x \cos x} = \log_e \tan x,$$

$$\int \frac{dx}{\sin x} = \log_e \tan \frac{x}{2}, \qquad \int \frac{dx}{\cos x} = \log_e \tan\left(\frac{\pi}{4} + \frac{x}{2}\right).$$

For $\int \sin^m x.dx$, use **1130**, putting $n = 0$.

$\int \cos^n x.dx$, use **1131**, putting $m = 0$.

$\int \tan^m x.dx$, use **1134**, putting $n = m$.

$\int \operatorname{ctn}^n x.dx$, use **1136**, putting $m = n$.

$\int \sec^n x.dx$, use **1137**, putting $m = 0$.

$\int \csc^m x.dx$, use **1138**, putting $n = 0$.

*Miscellaneous Integrals.*

**1139.** $$\int \frac{dx}{a \pm b \sin x} = \frac{2 \sec \beta}{a} \tan^{-1}\left(\sec \beta \tan \frac{x}{2} \pm \tan \beta\right) + C,$$
when $a > b$, and $b = a \sin \beta$.

**1140.** $$\int \frac{dx}{a \pm b \sin x} = \pm \frac{\sec \alpha}{b} \log_e \frac{\sin \frac{1}{2}(\alpha \pm x)}{\cos \frac{1}{2}(x \mp \alpha)} + C,$$
when $b > a$, and $a = b \sin \alpha$.

**1141.** $$\int \frac{dx}{a + b \cos x} = \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1}\left(\sqrt{\frac{a-b}{a+b}} \tan \frac{x}{2}\right) + C,$$
when $a^2 > b^2$.

**1142.** $$\int \frac{dx}{a + b \cos x} = \frac{2}{\sqrt{b^2 - a^2}} \operatorname{Tanh}^{-1}\left(\sqrt{\frac{b-a}{b+a}} \tan \frac{x}{2}\right) + C,$$
when $b^2 > a^2$.

**1143.** $$\int \frac{\cos x \, dx}{a + b \cos x} = \frac{x}{b} - \frac{a}{b} \int \frac{dx}{a + b \cos x} + C.$$

**1144.** $$\int \frac{\sin x \, dx}{a + b \cos x} = -\frac{1}{b} \log_e (a + b \cos x) + C.$$

1145. $\int \frac{dx}{a + b \tan x}$

$$= \frac{1}{a^2 + b^2}[b \log_e (a \cos x + b \sin x) + ax] + C.$$

1146. $\int \frac{dx}{\sin x + \cos x} = \frac{1}{\sqrt{2}} \log_e \tan (\frac{1}{2}x + \frac{1}{8}\pi).$

1147. $\int \sin mx \sin nx \, dx$

$$= \frac{\sin (m - n) x}{2 (m - n)} - \frac{\sin (m + n) x}{2 (m + n)} + C.$$

1148. $\int \sin mx \cos nx \, dx$

$$= - \frac{\cos (m - n) x}{2 (m - n)} - \frac{\cos (m + n) x}{2 (m + n)} + C.$$

1149. $\int \cos mx \cos nx \, dx$

$$= \frac{\sin (m - n) x}{2 (m - n)} + \frac{\sin (m + n) x}{2 (m + n)} + C.$$

1150. $\int x^n e^{ax} \, dx$

$$= \frac{x^n e^{ax}}{a}\left(1 - \frac{n}{ax} + \frac{n (n-1)}{a^2x^2} - \cdots \pm \frac{n!}{a^n x^n}\right) + C.$$

1151. $\int \log_e x \, dx = x \log_e x - x + C.$

1152. $\int (\log_e x)^n \, dx = x (\log_e x)^n - n \int (\log_e x)^{n-1} \, dx.$

1153. $\int x^m \log_e x \, dx = x^{m+1}\left[\frac{\log_e x}{m + 1} - \frac{1}{(m + 1)^2}\right] + C.$

1154. $\int x^m (\log_e x)^n \, dx$

$$= \frac{x^{m+1} (\log_e x)^n}{m + 1} - \frac{n}{m + 1} \int x^m (\log_e x)^{n-1} \, dx + C.$$

1155. $\int \frac{\log_e x}{x^2} dx = - \frac{\log_e x}{x} - \frac{1}{x} + C.$

1156. $\int \frac{(\log_e x)^n}{x} dx = \frac{1}{n + 1} (\log_e x)^{n+1} + C.$

*Definite Integrals.*

1157. $\int_a^b = -\int_b^a.$

1158. $\int_a^c = \int_a^b + \int_b^c.$

1159. $\int_a^c - \int_a^b = \int_b^c.$

1160. $\int_0^\infty \frac{dx}{a+bx^2} = \frac{\pi}{2\sqrt{ab}}.$

1161. $\int_0^{\sqrt{a/b}} \frac{dx}{a+bx^2} = \int_{\sqrt{a/b}}^\infty \frac{dx}{a+bx^2} = \frac{\pi}{4\sqrt{ab}}.$

1162. $\int_0^{\sqrt{a/b}} \frac{dx}{\sqrt{a-bx^2}} = \frac{\pi}{2\sqrt{b}}.$

1163. $\int_0^\infty \frac{\sin bx}{x}\,dx = \frac{\pi}{2},$ if $b$ is positive; $= -\frac{\pi}{2},$ if $b$ is negative; $=0$, if $b = 0$.

1164. $\int_0^\infty \frac{\cos bx}{x}\,dx = \infty.$

1165. $\int_0^\infty \frac{\sin x\,dx}{\sqrt{x}} = \int_0^\infty \frac{\cos x\,dx}{\sqrt{x}} = \sqrt{\tfrac{1}{2}\pi}.$

1166. $\int_0^\pi \sin^2 mx\,dx = \int_0^\pi \cos^2 mx\,dx = \tfrac{1}{2}\pi.$

1167. $\int_0^{\frac{1}{2}\pi} \sin^{2n+1} x\,dx = \int_0^{\frac{1}{2}\pi} \cos^{2n+1} x\,dx,$

$$= \frac{2\times 4\times 6\times \ldots 2n}{3\times 5\times 7\times \ldots (2n+1)}.$$

1168. $\int_0^{\frac{1}{2}\pi} \sin^{2n} x\,dx = \int_0^{\frac{1}{2}\pi} \cos^{2n} x\,dx,$

$$= \frac{1\times 3\times 5\times \ldots (2n-1)}{2\times 4\times 6\times \ldots 2n}\cdot\frac{\pi}{2}.$$

(1167, 1168) wherein $n$ is a positive integral number.

**1169.** $\int_0^\infty e^{-x}\,dx = 1.$ **1170.** $\int_0^\infty e^{-x^2}\,dx = \tfrac{1}{2}\sqrt{\pi}.$

**1171.** $\int_0^\infty x^n e^{-ax}\,dx = \dfrac{n!}{a^{n+1}},$ [wherein $a$ is positive, and $n$ is a positive integer.

**1172.** $\int_0^\infty \dfrac{x^{n-1}\,dx}{x+1} = \dfrac{\pi}{\sin n\pi},$ [$0 < n < 1.$

*Approximate Integration.*

To find the area of the plane space included between an arc of the curve $y = f(x)$, the axis of $x$, and two fixed ordinates, suppose the given space to be divided into $n$ narrow strips by a series of equidistant ordinates, $h$ being their common distance apart. The larger the number of strips the narrower each strip, and the closer the approximation to the required area made by any of the following methods.

(i) Each strip regarded as a rectangle of base $h$ and height equal to the ordinate on the left-hand side of it. The areas of the strips are

$$hy_0,\ hy_1,\ hy_2,\ \ldots\ hy_{n-1},$$

and their sum, $A$, is approximately the area of the given space.

**1181.** $A = h\,(y_0 + y_1 + y_2 + \ldots + y_{n-1}).$

(ii) Each strip regarded as a trapezoid the areas of which are

$$h\,\frac{y_0+y_1}{2},\ h\,\frac{y_1+y_2}{2},\ h\,\frac{y_2+y_3}{2},\ \ldots\ h\,\frac{y_{n-1}+y_n}{2}$$

and their sum

**1182.** $A = h\,(\tfrac{1}{2}y_0 + y_1 + y_2 + \ldots + y_{n-1} + \tfrac{1}{2}y_n).$

This approximation is closer than that given by **1181.**

(iii) A still closer approximation is obtained by regarding the curve passing through the ends of each set of three

successive ordinates as being approximately the arc of a parabola having its axis parallel to the ordinates. For this purpose, make $n$ an even number.

The area of the two strips lying between the ordinates $y_0, y_1, y_2$ is

$$\tfrac{1}{3}h\,(y_0 + 4y_1 + y_2).$$

The area of the next two strips lying between the ordinates $y_2, y_3, y_4$ is

$$\tfrac{1}{3}h\,(y_2 + 4y_3 + y_4).$$

And so on, the area of the last two strips being

$$\tfrac{1}{3}h\,(y_{n-2} + 4y_{n-1} + y_n).$$

The sum of these $\tfrac{1}{2}n$ double strips is

1183. $$A = \tfrac{1}{3}h\,[y_0 + 4\,(y_1 + y_3 + y_5 + \ldots + y_{n-1}) + 2\,(y_2 + y_4 + y_6 + \ldots + y_{n-2}) + y_n].$$

This formula is known under the name of *Simpson's Rule*.

Simpson's Rule may be used for computing the value of the definite integral

$$\int_a^b f\,(x)\,dx,$$

provided $f\,(x)$ and its derivatives $f'\,(x)$, $f''\,(x)$, and $f'''\,(x)$ remain finite and continuous in the interval from $x = a$ to $x = b$, and provided further that $f^{\text{iv}}\,(x)$ undergo no change of sign in the same interval. Under these conditions

1184. $$\int_a^b f\,(x)\,dx = \tfrac{1}{3}h\,[y_0 + 4\,(y_1 + y_3 + y_5 + \ldots + y_{n-1}) + 2\,(y_2 + y_4 + y_6 + \ldots + y_{n-2}) + y_n] + \tfrac{1}{288}\,\rho\,h_4\,[f'''\,(b) - f'''\,(a)].$$

The last term represents $E$, the error of Simpson's Rule, that is

1185. $$E = \tfrac{1}{288}\,\rho\,h^4\,[f'''\,(b) - f'''\,(a)].$$

In this expression $\rho$ is no further known than that it is a positive or negative proper fraction,

$$-1 < \rho < +1;$$

therefore the limits of the error are found by substituting $-1$ and $+1$ for $\rho$ in the above expression.

1186. Limits of error $= \pm \frac{h^4}{288}[f'''(b) - f'''(a)]$.

*Differential Equations of the First Order.*

A differential equation of the first order involves only the first derivatives of the dependent variable.

1201. Given, $f(x)\,dx + F(y)\,dy = 0$.

Solution, $$\int f(x)\,dx + \int F(y)\,dy = C.$$

*Separation of the Variables.*

1202. Given, $f(x).\varphi(y)\,dx + F(x).\psi(y)\,dy = 0$.

Solution, $$\int \frac{f(x)}{F(x)}\,dx + \int \frac{\psi(y)}{\varphi(y)}\,dy = C.$$

1203. Given, $f(x, y)\,dx + \varphi(x, y)\,dy = 0$.
This form is integrable if

$$D_y f(x, y) = D_x \varphi(x, y);$$

and the solution is either

$$\int f(x, y)\,dx + \int \left[\varphi(x, y) - \int D_y f(x, y) dx\right] dy = C,$$

or $$\int \varphi(x, y)\,dy + \int \left[f(x, y) - \int D_x \varphi(x, y) dy\right] dx = C.$$

1204. But if the condition of integrability above given is not satisfied, the equation may become integrable when multiplied by an integrating factor, $M$, which must satisfy the partial differential equation

$$f(x, y)\,D_y M - \varphi(x, y)\,D_x M = M\,[D_x \varphi(x, y) - D_y f(x, y)].$$

1205. In many cases $M$ can be found from the equations

$$\frac{dy}{f(x, y)} = -\frac{dx}{\varphi(x, y)} = \frac{dM}{M[D_x\varphi(x, y) - D_y f(x, y)]}.$$

*Homogeneous Differential Equations.*

1206. If a differential equation of the first order, homogeneous as to $x$ and $y$, can be reduced to the general form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right),$$

it can be solved by putting

$$\frac{y}{x} = t, \; y = xt, \; dy = xdt + tdx.$$

The solution is

$$\log_e x = \int \frac{dt}{f(t) - t} + C.$$

*Linear Differential Equations.*

A linear differential equation is one in which the dependent variable and its derivatives occur in the first degree only. The general form of such an equation of the first order is

$$\frac{dy}{dx} + Py = Q,$$

wherein $P$ and $Q$ are functions of $x$ or constants.

1207. The method of solution is as follows.

First, determine $y$ by integrating the equation

$$\frac{dy}{dx} + Py = 0, \text{ that is, } \frac{dy}{y} = -Pdx,$$

which gives

$$y = Ce^{-\int Pdx}, \text{ or } y\,e^{\int Pdx} = C.$$

Next, differentiate $y\,e^{\int Pdx} = C$, obtaining

$$e^{\int Pdx}(dy + Py\,dx) = 0;$$

which shows that $e^{\int Pdx}$ is an integrating factor of the given equation.

Finally, multiply the given equation by this integrating factor, obtaining

$$e^{\int P dx}(dy + Py\,dx) = e^{\int P dx}\,Qdx.$$

which integrated gives

$$ye^{\int P dx} = \int e^{\int P dx}\,Qdx + C,$$

or $$y = e^{-\int P dx}\left[\int e^{\int P dx}\,Qdx + C\right].$$

1208. A differential equation of the general form

$$\frac{dy}{dx} + Py = Qy^n,$$

wherein $P$ and $Q$ are functions of $x$ or constants, can be reduced to the linear form by dividing through by $y^n$, multiplying through by $(-n+1)$, and, in the result

$$(-n+1)\,y^{-n}\frac{dy}{dx} + (-n+1)\,Py^{-n+1} = (-n+1)\,Q,$$

putting $z$ for $y^{-n+1}$. The result is thus reduced to

$$\frac{dz}{dx} + (1-n)\,Pz = (1-n)\,Q,$$

which is linear as to $z$.

*Differential Equations of the Second Order.*

1209. Given $$\frac{d^2y}{dx^2} = f(x).$$

Solutions

(1) $$y = \int dx \int f(x)\,dx + C_1x + C_2.$$

(2) $$y = x\int f(x)\,dx - \int xf(x)\,dx + C_1x + C_2.$$

1210. Given

$$\frac{d^2y}{dx^2} = f(y).$$

Solution

$$x = \int \frac{dy}{\sqrt{C_1 + 2\int f(y)dy}} + C_2.$$

1211. Given

$$\frac{d^2y}{dx^2} = f\left(\frac{dy}{dx}\right).$$

By putting $\frac{dy}{dx} = z$, $\frac{d^2y}{dx^2} = \frac{dz}{dx}$, there are derived two equations,

$$x = \int \frac{dz}{f(z)} + C_1, \qquad y = \int \frac{zdz}{f(z)} + C_2,$$

from which, by the elimination of $z$, the required solution results.

1212. Given

$$\frac{d^2y}{dx^2} = f\left(\frac{dy}{dx}, x\right).$$

By putting $\frac{dy}{dx} = z$, $\frac{d^2y}{dx^2} = \frac{dz}{dx}$, in the given equation, the result is a differential equation of the first order in $z$, namely $\frac{dz}{dx} = f(z, x)$, which may be integrable.

If the result of integrating it be $z = \varphi(x)$, the solution of the given equation is then obtained by integrating $dy = \varphi(x)\, dx$.

1213. Given $\frac{d^2y}{dx} = f\left(\frac{dy}{dx}, y\right).$

By putting $\frac{dy}{dx} = z$, $\frac{d^2y}{dx^2} = z\frac{dz}{dy}$, in the given equation,

the result is a differential equation of the first order in $z$, namely

$$z \frac{dz}{dy} = f(z, y),$$

which may be integrable. If the result of integrating it be

$$z = \varphi(y),$$

then the integral of the given equation is

$$x = \int \frac{dy}{\varphi(y)} + C.$$

*Differential Equations of the $n^{th}$ order with constant coefficients.*

Given

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_{n-1} \frac{dy}{dx} + a_n y = 0.$$

(i) The substitution of $e^{rx}$ for $y$ in each term of this equation gives

$$(a_0 r^n + a_1 r^{n-1} + a_2 r^{n-2} + \ldots + a_n)\, e^{rx},$$

an expression which vanishes for every value of $r$ which satisfies the equation

$$a_0 r^n + a_1 r^{n-1} + a_2 r^{n-2} + \ldots + a_n = 0.$$

If $r_1, r_2, \ldots r_n$ are the roots of this equation, the general solution of the given equation is

1214. $$y = C_1 e^{r_1 x} + C_2 e^{r_2 x} + C_3 e^{r_3 x} + \ldots + C_n e^{r_n x},$$

and each term in the right-hand member of this equation is a particular solution of the given equation.

(ii) If there is one imaginary root, say $r_3 = p + iq$, there must be another, its conjugate, say $r_4 = p - iq$, provided the coefficients $a_0, a_1, a_2, \ldots a_n$ are all real. The general solution then becomes

1215. $$y = C_1 e^{r_1 x} + C_2 e^{r_2 x} + e^{px} [C_3 \cos qx + C_4 \sin qx] + C_5 e^{r_5 x} + \ldots + C_n e^{r_n x}.$$

(iii) If some, say $p$, of the roots are equal,

$$r_1 = r_2 = r_3 = \dots = r_p,$$

the general solution then becomes

1216. $$y = e^{r_1 x}(C_1 + C_2 x + C_3 x^2 + \dots + C_p x^{p-1}) + C_{p+1} e^{r_{p+1} x} + \dots + C_n e^{r_n x}.$$

1217. Given

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = f(x).$$

The roots $r_1, r_2, r_3, \dots r_n$ are first determined as in the last problem.

The general solution is

$$y = z_1 e^{r_1 x} + z_2 e^{r_2 x} + z_3 e^{r_3 x} + \dots + z_n e^{r_n x},$$

wherein $z_1, z_2, z_3, \dots z_n$ are the results of solving the $n$ equations

$$e^{r_1 x} \frac{dz_1}{dx} + e^{r_2 x} \frac{dz_2}{dx} + e^{r_3 x} \frac{dz_3}{dx} + \cdots + e^{r_n x} \frac{dz_n}{dx} = 0.$$

$$r_1 e^{r_1 x} \frac{dz_1}{dx} + r_2 e^{r_2 x} \frac{dz_2}{dx} + r_3 e^{r_3 x} \frac{dz_3}{dx} + \cdots + r_n e^{r_n x} \frac{dz_n}{dx} = 0.$$

$$r_1^2 e^{r_1 x} \frac{dz_1}{dx} + r_2^2 e^{r_2 x} \frac{dz_2}{dx} + r_3^2 e^{r_3 x} \frac{dz_3}{dx} + \cdots + r_n^2 e^{r_n x} \frac{dz_n}{dx} = 0$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$r_1^{n-1} e^{r_1 x} \frac{dz_1}{dx} + r_2^{n-1} e^{r_2 x} \frac{dz_2}{dx} + \dots + r_n^{n-1} e^{r_n x} \frac{dz_n}{dx} = f(x).$$

If the given equation had a constant $A$ instead of $f(x)$ for its right-hand member, its solution would be

1218. $$y = C_1 e^{r_1 x} + C_2 e^{r_2 x} + \dots + C_n e^{r_n x} + \frac{A}{a_n}.$$

# SECTION V.

## ANALYTIC GEOMETRY.

### *The Point and the Straight Line in a Plane.*

N.B. Rectangular coördinates are assumed unless otherwise stated.

The coördinates of a point $(x, y)$ dividing the distance between two points $(x_1, y_1)$ and $(x_2, y_2)$ in a given ratio $l : m$ are

1301. $$x = \frac{lx_2 + mx_1}{l + m}, \qquad y = \frac{ly_2 + my_1}{l + m}.$$

The point of bisection is given by

1302. $$x = \frac{x_2 + x_1}{2}, \qquad y = \frac{y_2 + y_1}{2}.$$

The point of external division is given by

1303. $$x = \frac{lx_2 - mx_1}{l - m}, \qquad y = \frac{ly_2 - my_1}{l - m}.$$

The distance between two points $(x_1, y_1)$ and $(x_2, y_2)$ is

1304. $$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

1305. If the axes be oblique making an angle $\omega$ the distance is given by the equation

$$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + 2\,(x_1 - x_2)\,(y_1 - y_2)\cos\omega.$$

If the angle which the line $d$ makes with the axis of $x$ is $\alpha$, then

1306. $$\cos\alpha = \frac{x_1 - x_2}{d}, \qquad \sin\alpha = \frac{y_1 - y_2}{d}.$$

The distance of any point $(x, y)$ from the origin is

1307. $$d = \sqrt{x^2 + y^2}.$$

If the axes be oblique, making an angle $\omega$,

1308. $$d = \sqrt{x^2 + y^2 + 2xy \cos \omega}.$$

The general equation of the straight line is

1309. $$Ax + By + C = 0.$$

If $A = 0$, the straight line is parallel to the axis of $x$.
If $B = 0$, the straight line is parallel to the axis of $y$.
If $C = 0$, the straight line passes through the origin.
If $a$ and $b$ be the intercepts which the straight line makes on the axes of $x$ and $y$, the equation may be written

1310. $$\frac{x}{a} + \frac{y}{b} = 1. \quad \left[a = \frac{-C}{A},\ b = \frac{-C}{B}\right.$$

If the straight line makes an angle $\beta$ with the positive direction of the axis of $x$ and makes an intercept $b$ on the axis of $y$, its equation is

1311. $$y = mx + b,$$

wherein $m = \tan \beta$. The coefficient $m$ is often called the *slope* of the line.

$$\left[m = \frac{-A}{B} = -\frac{b}{a} = \tan \beta.\right.$$

If the straight line passes through a given point $(x_1, y_1)$ and has a given inclination to the axis of $x$, its equation is

1312. $$y - y_1 = m(x - x_1).$$

If the straight line passes through two given points $(x_1, y_1)$ and $(x_2, y_2)$ its equation is

1313. $$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}.$$

$$\left[m = \tan \beta = \frac{y_2 - y_1}{x_2 - x_1}.\right.$$

If a perpendicular from the origin upon a straight line be $p$ in length and make an angle $\alpha$ with the axis of $x$ the equation of the straight line takes the *normal* form,

1314. $$x \cos \alpha + y \sin \alpha - p = 0.$$

An equation of the form $Ax + By + C = 0$ is reduced to the normal form by putting

1315. $$\frac{A}{\pm\sqrt{A^2+B^2}} = \cos\alpha, \quad \frac{B}{\pm\sqrt{A^2+B^2}} = \sin\alpha,$$
$$\frac{-C}{\pm\sqrt{A^2+B^2}} = p.$$

N.B. The sign of the radical is to be so chosen as to give $p$ a positive value.

The length of the perpendicular drawn from any point $(x_1, y_1)$ to the straight line $x\cos\alpha + y\sin\alpha - p = 0$ is

1316. $$l = \pm(x_1\cos\alpha + y_1\sin\alpha - p),$$

wherein the positive sign is to be taken when the point $(x_1, y_1)$ and the origin are on opposite sides of the line, and the negative sign when they are on the same side of the line.

If the equation of the given straight line be of the form $Ax + By + C = 0$, the length of the perpendicular is given in the form

1317. $$l = \frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}}.$$

All points for which the numerator of this fraction has the same sign as $C$ are on the same side of the given line as the origin.

The equation

1318. $$(Ax + By + C) + k(A'x + B'y + C') = 0$$

wherein $k$ is arbitrary, represents all straight lines drawn through the intersection of the two straight lines

$$Ax + By + C = 0 \quad \text{and} \quad A'x + B'y + C' = 0$$

The bisectors of the supplementary angles between

$$Ax + By + C = 0 \quad \text{and} \quad A'x + B'y + C' = 0$$

are represented by the equation

1319. $$\frac{Ax + By + C}{\sqrt{A^2 + B^2}} \mp \frac{A'x + B'y + C'}{\sqrt{A'^2 + B'^2}} = 0.$$

Or, if the equations of the given straight lines are in the normal form **1314**, the equations of the bisectors are

$$(x \cos \alpha + y \sin \alpha - p) \pm (x \cos \alpha' + y \sin \alpha' - p') = 0.$$

The condition that three straight lines pass through one and the same point is that the determinant formed by eliminating $x$ and $y$ from their three equations should vanish, that is

**1320.** $$\begin{vmatrix} A & B & C \\ A' & B' & C' \\ A'' & B'' & C'' \end{vmatrix} = 0.$$

**1321.** The condition that three points $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ lie on one straight line is

$$y_1 (x_2 - x_3) + y_2 (x_3 - x_1) + y_3 (x_1 - x_2) = 0,$$

that is, that the determinant

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

The angle $\theta$ which two straight lines

$$Ax + By + C = 0 \text{ and } A'x + B'y + C' = 0$$

make with each other is given by either of the equations

**1322.** $$\cos \theta = \frac{AA' + BB'}{\pm\sqrt{(A^2 + B^2)}\sqrt{(A'^2 + B'^2)}}.$$

**1323.** $$\sin \theta = \frac{A'B - AB'}{\pm \sqrt{A^2 + B^2}\sqrt{A'^2 + B'^2}}.$$

**1324.** $$\tan \theta = \frac{A'B - AB'}{AA' + BB'}.$$

The lines are perpendicular to each other ($\theta = 90°$) when

**1325.** $$AA' + BB' = 0;$$

they are parallel ($\theta = 0°$) when

**1326.** $$\frac{A}{A'} = \frac{B}{B'}.$$

If the equations of the straight lines are given in the explicit form

$$y = mx + b \text{ and } y = m'x + b'$$

the angle which they make with each other is given by the equation

1327. $$\tan\theta = \frac{m - m'}{1 + m'm}.$$

1328. The lines are parallel when $m' = m$.

1329. They are perpendicular when $m'm = -1$.

The equation of a straight line which cuts a given straight line $y = mx + b$ in the point $(x_1, y_1)$ and makes with it an angle $\theta$ is

1330. $$y - y_1 = \frac{m + \tan\theta}{1 - m\tan\theta}(x - x_1).$$

The equation of a straight line passing through the point $(x_1, y_1)$ and perpendicular to the given line $y = mx + b$ is

1331. $$y - y_1 = -\frac{1}{m}(x - x_1).$$

Or, if the given straight line is represented by the equation $Ax + By + C = 0$, the equation of the perpendicular through $(x_1, y_1)$ is

1332. $$A(y - y_1) = B(x - x_1).$$

Two straight lines through the point $(x_1, y_1)$ and making an angle $\theta$ with the line $y = mx + b$

1333. $$\frac{y - y_1}{x - x_1} = \frac{m - \tan\theta}{1 + m\tan\theta}, \quad \frac{y - y_1}{x - x_1} = \frac{m + \tan\theta}{1 - m\tan\theta}.$$

1334. Double the area of the triangle the vertices of which are the three points $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ is the determinant

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

The condition that these three points lie in one straight line is that this determinant vanish. Hence the equation of a straight line passing through two given points may be written

1335.
$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

A system of polar coördinates of a point in a plane consists of

$r =$ the distance of the point from the origin.
$\varphi =$ the angle which $r$ makes with the positive direction of the axis of $x$.

These polar coördinates are related to rectangular coördinates as follows:

1336. $r = \sqrt{x^2 + y^2}$, $x = r \cos \varphi$, $y = r \sin \varphi$, $\varphi = \tan^{-1} \frac{y}{x}$.

The distance, $d$, between two points $(r_1, \varphi_1)$ and $(r_2, \varphi_2)$ is given by the equation

1337. $$d^2 = r_1^2 + r_2^2 - 2r_1r_2 \cos (\varphi_1 - \varphi_2).$$

The polar equation of a straight line is

1338. $$r \cos (\varphi - \alpha) = p,$$

wherein $p$ is the perpendicular from the origin upon the straight line and $\alpha$ is the angle which $p$ makes with the fixed axis.

The polar equation of a straight line passing through two points $(r_1, \varphi_1)$ and $(r_2, \varphi_2)$ is

1339.
$$rr_1 \sin (\varphi - \varphi_1) + r_1r_2 \sin (\varphi_1 - \varphi_2) + r_2r \sin (\varphi_2 - \varphi) = 0.$$

*Transformation of Coördinates.*

If $x$ and $y$ be the coördinates of any point, and $x'$ and $y'$ the coördinates of the same point with reference to a new set of axes, transformation of an equation to the new axes

is effected by substituting in the equation for $x$ and $y$ their respective equivalents in terms of the new coördinates $x'$ and $y'$. Let $x_0$ and $y_0$ be the coördinates of the new origin referred to the old axes. Let the angle between the positive parts of any two axes $a$ and $a'$ be expressed by the notation $\genfrac{}{}{0pt}{}{a'}{a}$ or $\genfrac{}{}{0pt}{}{a}{a'}$, the one being the negative of the other, and the reading in every case being from the lower letter to the upper. Thus $\sin \genfrac{}{}{0pt}{}{a'}{a}$ means the sine of the angle through which $a$ must turn to take the direction of $a'$.

With this notation the formulas for transformation are written as follows:

**1340.** New axes parallel to the old. $\genfrac{}{}{0pt}{}{x'}{x} = 0,\ \genfrac{}{}{0pt}{}{y'}{y} = 0.$

$$x = x_0 + x',$$
$$y = y_0 + y'.$$

These formulas hold for both oblique and rectangular axes.

**1341.** From oblique axes to oblique.

$$(x - x_0)\sin \genfrac{}{}{0pt}{}{x}{y} = x' \sin \genfrac{}{}{0pt}{}{x'}{y} + y' \sin \genfrac{}{}{0pt}{}{y'}{y},$$
$$(y - y_0)\sin \genfrac{}{}{0pt}{}{y}{x} = x' \sin \genfrac{}{}{0pt}{}{x'}{x} + y' \sin \genfrac{}{}{0pt}{}{y'}{x}.$$

**1342.** From oblique axes to rectangular.

$$(x - x_0)\sin \genfrac{}{}{0pt}{}{x}{y} = x' \sin \genfrac{}{}{0pt}{}{x'}{y} + y' \cos \genfrac{}{}{0pt}{}{x'}{y},$$
$$(y - y_0)\sin \genfrac{}{}{0pt}{}{y}{x} = x' \sin \genfrac{}{}{0pt}{}{x'}{x} + y' \cos \genfrac{}{}{0pt}{}{x'}{x},$$

**1343.** From oblique axes to rectangular, when the axis of $x'$ is parallel to that of $x$, that is $\genfrac{}{}{0pt}{}{x'}{x} = 0,\ \genfrac{}{}{0pt}{}{y'}{x} = 90°$.

$$(x - x_0)\sin \genfrac{}{}{0pt}{}{x}{y} = x' \sin \genfrac{}{}{0pt}{}{x}{y} + y' \cos \genfrac{}{}{0pt}{}{x}{y},$$
$$(y - y_0)\sin \genfrac{}{}{0pt}{}{y}{x} = y'.$$

1344. From oblique axes to rectangular, when the axis of $y'$ is parallel to that of $y$, that is $\genfrac{}{}{0pt}{}{y'}{y} = 0$, $\genfrac{}{}{0pt}{}{x'}{y} = 90°$ or $270°$.

$$(x - x_0)\sin\genfrac{}{}{0pt}{}{x}{y} = x',$$

$$(y - y_0)\sin\genfrac{}{}{0pt}{}{y}{x} = x'\sin\genfrac{}{}{0pt}{}{x'}{x} + y'\cos\genfrac{}{}{0pt}{}{x'}{x}.$$

1345. From rectangular axes to oblique.

$$x - x_0 = x'\cos\genfrac{}{}{0pt}{}{x'}{x} + y'\cos\genfrac{}{}{0pt}{}{y'}{x},$$

$$y - y_0 = x'\sin\genfrac{}{}{0pt}{}{x'}{x} + y'\sin\genfrac{}{}{0pt}{}{y'}{x}.$$

1346. From rectangular axes to oblique, when the axis of $x'$ is parallel to that of $x$, that is $\genfrac{}{}{0pt}{}{x'}{x} = 0$.

$$x - x_0 = x' + y'\cos\genfrac{}{}{0pt}{}{y'}{x},$$

$$y - y_0 = y'\sin\genfrac{}{}{0pt}{}{y'}{x}.$$

1347. Rectangular axes to oblique, when the axis of $y'$ is parallel to that of $y$, that is $\genfrac{}{}{0pt}{}{y'}{x} = 90°$.

$$x - x_0 = x'\cos\genfrac{}{}{0pt}{}{x'}{x},$$

$$y - y_0 = x'\sin\genfrac{}{}{0pt}{}{x'}{x} + y'.$$

1348. Rectangular axes to rectangular.

$$x - x_0 = x'\cos\genfrac{}{}{0pt}{}{x'}{x} - y'\sin\genfrac{}{}{0pt}{}{x'}{x},$$

$$y - y_0 = x'\sin\genfrac{}{}{0pt}{}{x'}{x} + y'\cos\genfrac{}{}{0pt}{}{x'}{x}.$$

By putting the angle $\genfrac{}{}{0pt}{}{x'}{x} = \genfrac{}{}{0pt}{}{y'}{y} = \theta$, these equations may be written in the form more commonly used,

1349.
$$x - x_0 = x'\cos\theta - y'\sin\theta,$$
$$y - y_0 = x'\sin\theta + y'\cos\theta.$$

*The general equation of the second degree* in two variables is

1350. $$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

The Discriminant is

1351. $$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2.$$

1352. The derivative relative to $c$ of the discriminant is

$$C = \begin{vmatrix} a & h \\ h & b \end{vmatrix} = ab - h^2.$$

1353. If $C$ be *not* 0, the curve represented is a *central* curve, either an ellipse (including the circle) or a hyperbola, according as $C$ is positive or negative.

1354. If $C = 0$, the curve represented is either a parabola or a pair of parallel straight lines.

1355. If $\Delta = 0$, the equation represents a pair of straight lines.

*Transformation of the general equation.*

If $C$ be *not* $= 0$, and therefore the curve be *central*, the general equation can be transformed to parallel axes through the center.

The coördinates $x_0$, $y_0$ of the center are found by solving the equations

$$ax_0 + hy_0 + g = 0,$$
$$hx_0 + by_0 + f = 0.$$

These equations express the condition that the new $f$ and $g$ in the transformed equation should each be equal to 0.

1356. The coördinates of the center are

$$x_0 = \frac{hf - bg}{ab - h^2}, \quad y_0 = \frac{hg - af}{ab - h^2}.$$

The general equation is thus freed of terms of the first degree and takes the form

1357. $$ax^2 + 2hxy + by^2 + c' = 0,$$

wherein $c'$ (which is the result of substituting the coördinates of the center for $x$ and $y$ in the general equation) $= \frac{\Delta}{C}$.

Further, the term in $xy$ can be made to disappear by turning the axes on the center so as to make the new $h$ vanish. This can be done by virtue of the relations that exist between the old and the new coefficients when an equation is transformed from one set of rectangular axes to another.

These relations in the present case are

1358. $$a + b = a' + b'$$
$$ab - h^2 = a'b' - h'^2$$

By making $h' = 0$, the new coefficients, $a'$, $b'$, become known; for their sum and their product are known. They are the two roots of a quadratic equation, which may be called *the discriminating quadratic:*

1359. $$k^2 - (a + b)\,k + ab - h^2 = 0.$$

The two values of $k$ found by solving this equation are the $a'$ and $b'$ sought.

The angle $\theta$ through which the axes are turned to make the term in $xy$ disappear is given by the equation

1360. $$\tan 2\theta = \frac{2h}{a - b}.$$

The angle $\theta$ is the inclination of the principal axis of the conic to the axis of $x$.

The general equation now becomes

1361. $$a'x^2 + b'y^2 + c' = 0.$$

This will assume different forms according to the signs of the roots of the discriminating quadratic.

(i) Let both roots be positive. The curve makes real intercepts on each axis, is a closed curve, and if $a$ and $b$ be these intercepts, the equation may be written in the form

1362. $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$ An Ellipse.

If $a = b$ the ellipse is a Circle.

(ii) Let one root be negative. Then

1363. $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$ A Hyperbola.

(iii) Let both roots be negative. Then

1364. $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = -1,$$ No real locus.

(iv) When $c' = \frac{\Delta}{C}$ vanishes in consequence of $\Delta = 0$,

1365. $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0.$$ An infinitely small ellipse at the origin. Or, a pair of imaginary straight lines intersecting at the origin.

1366. $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0.$$ A pair of real straight lines intersecting at the origin, being the asymptotes of the corresponding hyperbola.

If $C = 0$, that is if $ab - h^2 = 0$, the first three terms of the general equation form a perfect square. The coördinates of the center (see 1356) become infinite, so that transformation to parallel axes through the center is impossible.

By turning the axes without changing the origin through the angle $\theta$, given by $\tan 2\theta = \frac{2h}{a - b}$, the term in $xy$ can be made to disappear; and, since one root of the discriminating quadratic either $a'$ or $b'$ now becomes 0, the general equation also loses the term in $x^2$ or in $y^2$, and so takes the form

1367. $$b'y^2 + 2g'x + 2f'y + c' = 0.$$

This can be transformed to parallel axes through a new origin so as to make $c'$ and $f'$ disappear, so that finally the general equation becomes

1368. $$y^2 = px,$$ A Parabola.

If, in the general equation, $ab - h^2 = 0$, and also either $hf - bg = 0$ or $hg - af = 0$, the equation takes the form of a quadratic in $y$ or in $x$,

1369. $$b'y^2 + 2f'y + c' = 0,$$

which can be separated into factors of the form

1370. $$b'(y - y_1)(y - y_2) = 0$$

wherein $y_1$, $y_2$ are the roots of the quadratic. The equation under these conditions represents a pair of parallel straight lines, real, coincident, or imaginary.

## *SPECIAL FORMULAS.*

### *The Circle.*

The general equation of the circle is

1371. $$x^2 + y^2 + Ax + By + C = 0.$$

If the coördinates of the center be $a$, $b$ and the radius $r$ the equation is

1372. $$(x - a)^2 + (y - b)^2 = r^2.$$

If the center be at the origin, the equation, called the *central equation*, is

1373. $$x^2 + y^2 = r^2.$$

If the circumference of the circle pass through the origin, and the center be in the positive part of the axis of $x$, the equation is

1374. $$x^2 + y^2 = 2rx.$$

The equation of the circle referred to oblique axes is

1375. $$(x - a)^2 + (y - b)^2 + 2(x - a)(y - b)\cos\omega = r^2.$$

The polar equation of the circle in general form is

1376. $$\rho^2 + (A\cos\varphi + B\sin\varphi)\rho + C = 0.$$

The polar equation of a circle, when the fixed axis passes through the center and $d$ is the distance of the center from the pole, is

1377. $$\rho^2 - 2d\rho\cos\varphi + d^2 - r^2 = 0.$$

If the fixed axis does not pass through the center, but makes an angle $\alpha$ with the straight line that does pass through the center, the polar equation of the circle takes the form

1378. $$\rho^2 - 2d\rho\cos(\varphi - \alpha) + d^2 - r^2 = 0.$$

If the pole is on the circle, the equation becomes

1379. $$\rho = 2r\cos\varphi.$$

Two straight lines

$$y - y' = m_1(x - x') \quad \text{and} \quad y - y' = m_2(x - x')$$

passing through $(x', y')$ are tangents to the circle $x^2 + y^2 = r^2$ if $m_1$ and $m_2$ are the two roots of the quadratic equation

1380. $$(x'^2 - r^2)m^2 - 2x'y'm + y'^2 - r^2 = 0.$$

The equation of a circle being $(x-a)^2 + (y-b)^2 = r^2$, that of a tangent at $(x', y')$ and that of the polar to $(x', y')$ is

1381. $$(x-a)(x'-a) + (y-b)(y'-b) = r^2,$$

or $$(x-a)\cos\alpha + (y-b)\sin\alpha = r$$

wherein $\alpha$ is the angle between the axis of $x$ and the radius to or through the point $(x', y')$.

The equation of the chord joining two points $(x_1, y_1)$, $(x_2, y_2)$ on the circle $x^2 + y^2 = r^2$ is

1382. $$x(x_1 + x_2) + y(y_1 + y_2) = x_1x_2 + y_1y_2 + r^2,$$

or $$x\cos\tfrac{1}{2}(\theta_1 + \theta_2) + y\sin\tfrac{1}{2}(\theta_1 + \theta_2) = r\cos\tfrac{1}{2}(\theta_1 - \theta_2),$$

wherein $$r\cos\theta_1 = x_1, \quad r\sin\theta_1 = y_1, \text{ etc.}$$

The equation of a circle passing through three given points $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ is

1383. $$(x^2+y^2)\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} - (x_1^2+y_1^2)\begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x & y & 1 \end{vmatrix}$$

$$+ (x_2^2+y_2^2)\begin{vmatrix} x_3 & y_3 & 1 \\ x & y & 1 \\ x_1 & y_1 & 1 \end{vmatrix} - (x_3^2+y_3^2)\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

1384. Centers of similitude for two circles lie in the straight line passing through their centers and divide it internally and externally in the ratio of their radii. Apply 1301 and 1303.

If there be given two circles

$$x^2 + y^2 + Ax + By + C = 0$$
$$x^2 + y^2 + A'x + B'y + C' = 0$$

the equation

1385. $$(A - A')\,x + (B - B')\,y + C - C' = 0$$

represents their common chord, if they intersect, or their radical axis, whether they intersect or not.

1386. Tangents to two circles drawn from any point of their radical axis are equal.

*Conic Sections.*

The locus of a point which moves so that its distance $\rho$ from a fixed point bears a constant ratio $e$ to its distance $x$ from a fixed straight line is a conic section and its equation is

1387. $$\rho = ex.$$

The fixed point is the *focus* and the fixed straight line is the *directrix* of the curve. Taking the directrix as the axis of $y$ and a line perpendicular thereto passing through the focus as the axis of $x$, and denoting the distance from the focus to the directrix by $d$, the equation of the locus in rectangular coördinates is

1388. $$\left\{\begin{array}{l} (x - d)^2 + y^2 = e^2x^2, \\ \text{or} \quad (1 - e^2)\,x^2 + y^2 - 2dx + d^2 = 0. \end{array}\right.$$

This represents an ellipse when $e < 1$, a parabola when $e = 1$, and a hyperbola when $e > 1$.

The ratio $e$ is called the *eccentricity*. For the circle $e = 0$, the focus is the center, but the directrix is infinitely distant.

If the origin be moved to the focus the equation becomes

1389. $$x^2 + y^2 = e^2\,(x + d)^2.$$

If the origin be moved to the point where the curve cuts the axis of $x$ between the focus and the directrix the equation takes the form

1390. $$(1 - e^2)\,x^2 + y^2 - 2edx = 0.$$

*The Ellipse.*

If the origin be changed to the center, the equation $x^2 + y^2 = e^2(x+d)^2$ becomes for the Ellipse

1391. $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

wherein

1392. $a =$ semi-axis major $= \dfrac{de}{1-e^2}$,

1393. $b =$ semi-axis minor $= \dfrac{de}{\sqrt{1-e^2}}$,

1394. $c =$ distance from focus to center $= \dfrac{de^2}{1-e^2}$.

1395. $b^2 = a^2(1-e^2)$, 1396. $e^2 = \dfrac{a^2-b^2}{a^2}$.

1397. $c = ae$, 1398. $a^2 - b^2 = c^2$.

1399. The distance from the directrix to the center

$$= d + c = \frac{a}{e} = \frac{a^2}{c}.$$

1400. The distances from the directrix to the vertices of the ellipse are $\dfrac{d}{1+e}$ and $\dfrac{d}{1-e}$.

1401. The distances from the focus to the vertices of the ellipse are $\dfrac{de}{1+e}$ and $\dfrac{de}{1-e}$.

1402. The *latus rectum*, or double the ordinate at the focus, also called the *parameter*,

$$p = 2de = 2a(1-e^2) = \frac{2b^2}{a}.$$

1403. The minor axis is a mean proportional between the major axis and the latus rectum.

$$2b = \sqrt{2ap}.$$

If two lines $r'$ and $r''$, called *focal radii*, be drawn one from each focus to any one point on the ellipse,

1404. $r' = a - ex$ and $r'' = a + ex$,

whence, $r' + r'' = 2a$,

which last equation expresses the fact that the sum of the distances of any point on the ellipse to the two foci is constant, and equal to the axis major.

*The Hyperbola.*

The general equation $x^2 + y^2 = e^2 (x + d)^2$, when the origin is changed from the focus to the center, becomes for the Hyperbola

1405. $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

wherein

1406. $a =$ half the transverse axis $= \dfrac{de}{e^2 - 1}$,

1407. $b =$ half the conjugate axis $= \dfrac{de}{\sqrt{e^2 - 1}}$.

1408. $c =$ distance from center to focus $= \dfrac{de^2}{e^2 - 1}$.

1409. $b^2 = a^2 (e^2 - 1)$. 1410. $e^2 = \dfrac{a^2 + b^2}{a^2}$.

1411. $c = ae$. 1412. $a^2 + b^2 = c^2$.

1413. The distance from the center to the directrix

$$= c - d = \frac{a}{e} = \frac{a^2}{c}.$$

1414. The distances from the directrix to the vertices of the hyperbola are $\dfrac{d}{e+1}$ and $\dfrac{d}{e-1}$.

1415. The distances from the focus to the vertices are

$$\frac{de}{e-1} \text{ and } \frac{de}{e+1}.$$

1416. The latus rectum, $p = 2de = 2a (e^2 - 1) = \dfrac{2b^2}{a}$.

1417. The conjugate axis is a mean proportional between the transverse axis and the latus rectum.

The focal radii of any point on the hyperbola are

1418. $$r' = ex - a \quad \text{and} \quad r'' = ex + a,$$

whence $$r'' - r' = 2a,$$

which expresses the fact that the difference of the distances of any point on the hyperbola from the foci is constant and equal to the transverse axis.

The equation of the asymptotes to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is}$$

1419. $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0,$$

or $$\frac{x}{a} + \frac{y}{b} = 0 \quad \text{and} \quad \frac{x}{a} - \frac{y}{b} = 0.$$

The equation of the hyperbola referred to its asymptotes as axes is

1420. $$xy = \tfrac{1}{4}(a^2 + b^2).$$

The equation of the tangent at $(x'y')$ or of the polar of the point $(x'y')$ wherever situated is

1421. $$yx' + xy' = \tfrac{1}{2}(a^2 + b^2).$$

The equation of the equilateral hyperbola, when referred to the principal axes, is

1422. $$x^2 - y^2 = a^2,$$

and when referred to the asymptotes, it is

1423. $$xy = \tfrac{1}{2}a^2.$$

The eccentricity of the equilateral hyperbola $= \sqrt{2}$.

1424. The equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ represents a hyperbola which is conjugate to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. It has foci in the axis of $y$; eccentricity $e = \frac{\sqrt{a^2 + b^2}}{b}$; directrices parallel to the axis of $x$ and at the distance $\pm \frac{b}{e}$ therefrom; and the same asymptotes as the hyperbola to which it is conjugate.

1425. Two conics $\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1$ and $\frac{x^2}{a'^2} \pm \frac{y^2}{b'^2} = 1$ are confocal if

$$a^2 - a'^2 = b^2 - b'^2;$$

or if, one being an ellipse and the other a hyperbola,

$$a^2 - a'^2 = b^2 + b'^2.$$

## *The Parabola.*

The general equation $x^2 + y^2 = e^2 (x + d)^2$, when the origin is changed from the focus to the vertex of the curve, becomes for the Parabola, since $e = 1$,

1426. $$y^2 = 2dx,$$

which is also written

1427. $$y^2 = 4ax \text{ and } y^2 = px.$$

1428. The distance from the directrix to the vertex of the parabola, which is equal to the distance from the vertex to the focus $\Big\} = a = \frac{1}{2}d = \frac{1}{4}p.$

1429. The latus rectum or parameter $= p = 2d = 4a.$

The equation of a parabola referred to a tangent and the diameter through the point of contact is

1430. $$y^2 = 4a'x,$$

wherein $a'$ is the distance from the focus to the point of contact.

The equation of each conic section referred to the transverse axis ($x$) and the tangent at the vertex ($y$).

1431. $$\begin{cases} y^2 = px - \frac{b^2}{a^2}x^2, & \text{Ellipse,} \\ y^2 = px, & \text{Parabola,} \\ y^2 = px + \frac{b^2}{a^2}x^2, & \text{Hyperbola.} \end{cases}$$

The polar equation of the Ellipse or of the Hyperbola, the pole being at the center and the axis on the positive part of the transverse axis, is

1432. $$\frac{1}{\rho^2} = \frac{\cos^2\varphi}{a^2} \pm \frac{\sin^2\varphi}{b^2}, \text{ or } \rho^2 = \pm\frac{b^2}{1 - e^2\cos^2\varphi}.$$

The polar equation of any conic section, the pole being at the focus and the axis on the transverse axis, is

1433. $$\rho = \frac{de}{1 \pm e\cos\varphi},$$

wherein the upper sign is to be used when the axis points towards the *nearest vertex* of the curve, and the lower sign when the axis points the other way.

For the Parabola, the polar equation above given may be changed to the form

1434. $$\rho\cos^2\tfrac{1}{2}\varphi = a, \quad \text{or} \quad \rho\sin^2\tfrac{1}{2}\varphi = a.$$

The equation

1435. $xx' + yy' = r^2$, for the Circle,

1436. $\dfrac{xx'}{a^2} + \dfrac{yy'}{b^2} = 1$, for the Ellipse,

1437. $\dfrac{xx'}{a^2} - \dfrac{yy'}{b^2} = 1$, for the Hyperbola,

1438. $yy' = 2a(x + x')$, for the Parabola,

represents a straight line which is

(i) a tangent to the curve at $(x', y')$, when $(x', y')$ is on the curve.

(ii) the polar of the point $(x', y')$ relatively to the curve, wherever $(x', y')$ may be.

(iii) the chord passing through the points of contact of a pair of tangents drawn to the curve from $(x', y')$, when $(x', y')$ is outside the curve.

(iv) the locus of the intersection of pairs of tangents drawn at the ends of all chords passing through $(x', y')$, wherever $(x', y')$ may be.

The equation of the tangent to a conic in terms of its slope $m$ ( = the trigonometric tangent of the angle $\tau$ which the tangent line makes with the axis of $x$) is

1439. $y = mx \pm r\sqrt{m^2+1}$ for the Circle,

1440. $y = mx \pm \sqrt{a^2m^2+b^2}$ for the Ellipse,

1441. $y = mx \pm \sqrt{a^2m^2-b^2}$ for the Hyperbola,

1442. $y = mx + \frac{a}{m}$ for the Parabola.

The condition that the ellipse or hyperbola $\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1$ be touched by the straight line $y = mx + g$ is that

1443. $$g^2 = a^2m^2 \pm b^2.$$

The condition that the straight line $Ax + By + C = 0$ touch the ellipse $b^2x^2 + a^2y^2 - a^2b^2 = 0$ is that

1444. $$A^2a^2 + B^2b^2 = C^2.$$

The equation of the tangent to an ellipse may be written in the normal form,

1445. $$x \cos \alpha + y \sin \alpha = p = \sqrt{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha}$$
$$= a\sqrt{1 - e^2 \sin^2 \alpha}$$

wherein $\alpha$ denotes the inclination of $p$ to the axis of $x$.

Two straight lines,

$$y - y' = m_1(x - x') \text{ and } y - y' = m_2(x - x'),$$

passing through the point $(x', y')$ are tangents to the ellipse or hyperbola

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1,$$

if $m_1$ and $m_2$ are the two roots of the quadratic equation

1446. $$(x'^2 - a^2)m^2 - 2x'y'm + y'^2 \pm b^2 = 0.$$

Two straight lines,

$$y = m_1 x + \frac{a}{m_1} \text{ and } y = m_2 x + \frac{a}{m_2},$$

passing through the point $(x', y')$ are tangents to the parabola $y^2 = 4ax$,
if $m_1$ and $m_2$ are the two roots of the quadratic equation

1447. $$x'm^2 - y'm + a = 0.$$

The normal line is perpendicular to the tangent and passes through the point of contact $(x', y')$. Its equation is

1448. $y = \frac{y'}{x'} x$ for the Circle,

1449. $y - y' = -\frac{y'}{2a}(x - x')$ for the Parabola,

1450. $y - y' = \frac{a^2 y'}{b^2 x'}(x - x')$ for the Ellipse,

1451. $y - y' = -\frac{a^2 y'}{b^2 x'}(x - x')$ for the Hyperbola.

The equation of the normal to a conic in terms of its slope $n$ ($=$ the trigonometric tangent of the angle $\nu$ which the normal line makes with the axis of $x$) is

1452. $y = nx - \frac{n(a^2 \pm b^2)}{\sqrt{a^2 \pm b^2 n^2}}$ for the Ellipse and Hyperbola,

wherein $$n = \tan \nu = \frac{a^2 y'}{\pm b^2 x'}.$$

1453. $y = nx - an(2 + n^2)$ for the Parabola,

wherein $$n = \tan \nu = -\frac{y'}{2a}.$$

1454. The equation $\frac{y - y'}{x - x'} = \frac{a^2 y'}{\pm b^2 x'}$ represents a straight line passing through the pole $(x', y')$ perpendicular to the polar. It becomes the normal when $(x', y')$ is on the curve the polar then becoming a tangent.

*Diameters.*

The locus of the middle points of a system of chords all parallel to the line $y = mx$ is a diameter, the equation of which is

1455. $y = -\frac{1}{m}x$ for the Circle,

1456. $y = \frac{2a}{m}$ for the Parabola,

1457. $y = -\frac{b^2}{a^2m}x$ for the Ellipse,

1458. $y = \frac{b^2}{a^2m}x$ for the Hyperbola.

The equation of the diameter which bisects the system of chords inclined at the angle $\theta$ to the axis of $x$ is

1459. $\frac{x \cos \theta}{a^2} + \frac{y \sin \theta}{b^2} = 0$ for the Ellipse,

1460. $\frac{x \cos \theta}{a^2} - \frac{y \sin \theta}{b^2} = 0$ for the Hyperbola.

1461. In the ellipse or hyperbola the relation

$$m_1 m_2 = \frac{\mp b^2}{a^2}$$

expresses these facts:

(i) that $y = m_2 x$ bisects all the chords represented by $y = m_1 x + c$, wherein $c$ is an arbitrary constant;

(ii) that $y = m_1 x$ and $y = m_2 x$ are conjugate diameters, each bisecting all chords drawn parallel to the other;

(iii) that $y - y' = m_1 (x - x')$ and $y + y' = m_2 (x + x')$, being supplemental chords drawn from the point $(x', y')$ are parallel respectively to the conjugate diameters $y = m_1 x$ and $y = m_2 x$;

(iv) that the tangent $y - y' = m_2 (x - x')$ at $(x', y')$, the end of a diameter $y = m_1 x$, is parallel to the conjugate of that diameter, $y = m_2 x$, and to the system of chords $y = m_2 x + c$.

The perpendicular from the center upon the tangent at $(x, y)$ is given by

1462. $$\frac{1}{p^2} = \frac{x^2}{a^4} + \frac{y^2}{b^4}.$$

The lengths of the perpendiculars $p'$ and $p''$ from the foci of the ellipse or hyperbola to the tangent at $(x, y)$,

1463. $$p' = b\sqrt{\frac{r'}{r''}}, \qquad p'' = b\sqrt{\frac{r''}{r'}}$$

wherein $r'$ and $r''$ are the focal radii of the point $(x, y)$.

1464. It follows that $p'p'' = b^2$.

Length of the perpendicular $p$ from the focus of the parabola to the tangent at $(x, y)$,

1465. $$p = \sqrt{ax + a^2} = \sqrt{aa'},$$ see 1430.

The equation of a chord through two points $(x_1, y_1)$ and $(x_2, y_2)$ on the ellipse is

1466. $$\frac{x(x_1 + x_2)}{a^2} + \frac{y(y_1 + y_2)}{b^2} = \frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} + 1.$$

Or, denoting the points by their eccentric angles $\alpha$, $\beta$, (see 1473) the equation of the chord is

1467. $$\frac{x}{a}\cos\tfrac{1}{2}(\alpha + \beta) + \frac{y}{b}\sin\tfrac{1}{2}(\alpha + \beta) = \cos\tfrac{1}{2}(\alpha - \beta).$$

The equation of the chord joining two points $(x_1, y_1)$ and $(x_2, y_2)$ on the parabola $y^2 = 4ax$ is

1468. $$y(y_1 + y_2) = y_1y_2 + 4ax.$$

If $a'$ and $b'$ be the conjugate semi-diameters of an ellipse or of a hyperbola and $\alpha$ and $\beta$ the angles which $a'$ and $b'$ respectively make with $a$, then

1469. $a^2 \pm b^2 = a'^2 \pm b'^2$, 1470. $ab = a'b'\sin(\beta - \alpha)$,

1471. $$\mp\frac{b^2}{a^2} = \tan\alpha\tan\beta;$$

and the equations of these curves referred to a pair of conjugate diameters as oblique coördinate axes are

1472. $$\frac{x^2}{a'^2} \pm \frac{y^2}{b'^2} = 1.$$

The equations

1473. $$x = a \cos u \quad \text{and} \quad y = b \sin u$$

represent an ellipse; and $u$ is the *eccentric angle* for any point $(x, y)$ on the ellipse. In Astronomy, $u$ is called the *eccentric anomaly.*

The equations

1474. $$x = a \sec u \quad \text{and} \quad y = b \tan u$$

represent a hyperbola, and $u$ is the eccentric angle for any point $(x, y)$ on the hyperbola.

The equation of a tangent to the ellipse at a point for which the eccentric angle is $u$ is

1475. $$\frac{x}{a} \cos u + \frac{y}{b} \sin u = 1.$$

The equation of a tangent to the hyperbola at a point for which the eccentric angle is $u$ is

1476. $$\frac{x}{a} \sec u - \frac{y}{b} \tan u = 1.$$

1477. Two diameters are conjugate if $u_1$ and $u_2$, the eccentric angles of their ends, satisfy the condition

$$u_1 \mp u_2 = \frac{\pi}{2}.$$

## GENERAL PROPERTIES OF PLANE CURVES.

*Tangents, Normals, Curvature, Evolutes, Involutes, Areas, Length of Arc, Envelopes, Pedal Curves, Trajectories.*

### *Tangents and Normals.*

If the tangent to a curve, $y = f(x)$, make an angle $\tau$ with the axis of $x$, then

1501. $$\tan \tau = \frac{dy}{dx} = f'(x) = y'.$$

Also,

1502. $$\cos \tau = \frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} = \frac{1}{\sqrt{1+y'^2}},$$

1503. $$\sin \tau = \frac{\frac{dy}{dx}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} = \frac{y'}{\sqrt{1+y'^2}},$$

Or,

1504. $$\cos \tau = \frac{dx}{\sqrt{dx^2+dy^2}},$$

1505. $$\sin \tau = \frac{dy}{\sqrt{dx^2+dy^2}},$$

Or, since the differential of the arc of the curve

1506. $$ds = \sqrt{dx^2 + dy^2},$$

1507. $$\cos \tau = \frac{dx}{ds}, \quad \sin \tau = \frac{dy}{ds}.$$

If the equation of a plane curve be given in the explicit form $y = f(x)$, the equation of the tangent at any point $(x, y)$ is

1508. $$Y - y = \frac{dy}{dx}(X - x), \text{ or } Y - y = f'(x).(X - x),$$

wherein $X$ and $Y$ denote the running coördinates of the tangent, and $x$ and $y$ those of the curve.

The equation of the normal at $(x, y)$ is

1509. $$Y - y = -\frac{dx}{dy}(X - x)$$

wherein $X$ and $Y$ denote the running coördinates of the normal, and $x$ and $y$ those of the curve.

If the equation of the curve be given in the implicit form $F(x, y) = 0$, the equation of the tangent is

1510. $$D_xF.(X - x) + D_yF.(Y - y) = 0,$$

and the equation of the normal is

1511. $$D_yF.(X - x) - D_xF.(Y - y) = 0.$$

*Asymptotes.*

The equation of the tangent. 1508, may be written in the form

1512. $$Y = Xf'(x) + f(x) - xf'(x).$$

If, when $x$ increases to $\infty$, $f'(x)$ approaches a definite limit $A$ and $f(x) - xf'(x)$ also approaches a definite limit $B$, then the tangent becomes an *Asymptote*, of which the equation is

1513. $$Y = AX + B.$$

*Applications to Particular Curves.*

The equation of the tangent to the circle $x^2 + y^2 = r^2$ at the point $(x, y)$ is

1514. $$Xx + Yy = r^2.$$

The equation of the normal at the same point is

1515. $$Xy - Yx = 0.$$

The equation of a pair of tangents from the point $(x_1, y_1)$ to the circle $x^2 + y^2 = r^2$ is

1516. $$(x_1^2 + y_1^2 - r^2)(X^2 + Y^2 - r^2) = (Xx_1 + Yy_1 - r^2)^2.$$

The equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $(xy)$ is

1517. $$\frac{Xx}{a^2} + \frac{Yy}{b^2} = 1.$$

The equation of the normal at the same point is

1518. $$\frac{X - x}{b^2x} - \frac{Y - y}{a^2y} = 0, \text{ or } \frac{a^2X}{x} - \frac{b^2Y}{y} = c^2.$$

The equation of the tangent to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ at the point } (x, y) \text{ is}$$

1519. $$\frac{Xx}{a^2} - \frac{Yy}{b^2} = 1.$$

The equation of the normal at the same point is

1520. $$\frac{X-x}{b^2x}+\frac{Y-y}{a^2y}=0, \text{ or } \frac{a^2X}{x}+\frac{b^2Y}{y}=c^2.$$

The equation of the tangent to the parabola $y^2=2dx$ at the point $(x, y)$ is

1521. $$Yy=d\,(X+x), \text{ or } \frac{Y}{\frac{1}{2}y}-\frac{X}{x}=1.$$

The equation of the normal at the same point is

1522. $$Y-y=-\frac{y}{d}\,(X-x).$$

The general equation of the tangent to the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

at the point $(x, y)$ is

1523.
$$aXx+h\,(Xy+Yx)+bYy+g\,(X+x)+f\,(Y+y)+c=0.$$

*The tangent* is the name given to that part of the tangent line which extends from T, where it cuts the axis of $x$, to P, where it touches the curve = TP.

1525. $$\text{TP}=y\frac{dx}{dy}\sqrt{1+\left(\frac{dy}{dx}\right)^2}=y\csc\tau=y\frac{ds}{dy}$$
$$=\frac{y}{y'}\sqrt{1+y'^2},$$

wherein $y'$ is written for $f'(x)$, derived from the equation

$$y=f\,(x).$$

The *subtangent* is the projection of the tangent on the axis of $x$.

1526. $$\text{TP}'=y\frac{dx}{dy}=y\operatorname{ctn}\tau=\frac{y}{y'}.$$

*The normal* is the name given to that part of the normal line which extends from N, where it cuts the axis of $x$, to P, where it cuts the curve $=$ NP.

1527. $NP = y\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = y \sec\tau = y\frac{ds}{dx} = y\sqrt{1 + y'^2}.$

The *subnormal* is the projection of the normal on the axis of $x$.

1528. $NP' = y\frac{dy}{dx} = y\tan\tau = yy'.$

Circle $x^2 + y^2 = r^2$,

1529. Tangent $= \frac{ry}{x}$. 1530. Normal $= r$.

1531. Subtangent $= \frac{-r^2}{x} + x$. 1532. Subnormal $= x$.

Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

1533. Tangent $= \frac{ay}{bx}\sqrt{a^2 - e^2x^2}$,

1534. Normal $= \frac{b}{a}\sqrt{a^2 - e^2x^2}$.

1535. Subtangent $= \frac{-a^2}{x} + x$.

1536. Subnormal $= \frac{-b^2x}{a^2}$.

Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$,

1537. Tangent $= \frac{ay}{bx}\sqrt{e^2x^2 - a^2}$,

1538. Normal $= \frac{b}{a}\sqrt{e^2x^2 - a^2}$,

1539. Subtangent $= \frac{a^2}{x} - x$, 1540. Subnormal $= \frac{b^2x}{a^2}$.

Parabola $y^2 = 4ax$,

1541. Subtangent $= 2x$,

1542. Subnormal $= 2a$ (constant).

*Polar Tangents and Normals.*

If $r$ denote the distance from the origin to P, the point where the tangent touches the curve, and $\varphi$ denote the angle which $r$ makes with the axis of $x$; and if through the origin a straight line be drawn perpendicular to $r$ and meeting the tangent and normal lines in $T_0$ and $N_0$ respectively; then

**1543.** Polar Tangent $= PT_0 = r\sqrt{1 + r^2\left(\frac{d\varphi}{dr}\right)^2} = \frac{rds}{dr}$.

**1544.** Polar Subtangent $= OT_0 = \frac{r^2 d\varphi}{dr} = \frac{r^2}{r'}$.

**1545.** Polar Normal $= PN_0 = \sqrt{r^2 + \left(\frac{dr}{d\varphi}\right)^2} = \frac{ds}{d\varphi}$

$$= \sqrt{r^2 + r'^2}.$$

**1546.** Polar Subnormal $= ON_0 = \frac{dr}{d\varphi} = r'$.

**1547.** $ds^2 = dr^2 + (rd\varphi)^2$, or $ds = d\varphi\sqrt{r^2 + r'^2}$.

In these and following formulas, $r'$, $r''$ are written for $f'(\varphi)$, $f''(\varphi)$, or for $\frac{dr}{d\varphi}$, $\frac{d^2r}{d\varphi^2}$, derived from the equation $r = f(\varphi)$.

The angle between the tangent and the radius vector is $\tau - \varphi$, and

**1548.** $$\operatorname{ctn}(\tau - \varphi) = \frac{r'}{r}.$$

*Curvature.*

The radius of curvature, or radius of the osculating circle,

**1549.** $$\rho = \frac{ds}{d\tau} = \frac{\left(\frac{ds}{dx}\right)^3}{\frac{d^2y}{dx^2}} = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{(1 + y'^2)^{\frac{3}{2}}}{y''},$$

wherein $y''$ is written for $f''(x)$.

Also

1550. $\rho = \dfrac{n^3}{y^3 y''}$, wherein $n =$ the normal.

The radius of curvature in polar coördinates is

1551. $$\rho = \frac{(r^2 + r'^2)^{\frac{3}{2}}}{r^2 + 2r'^2 - rr''}.$$

The coördinates $X_0$ and $Y_0$ of the centre of curvature are

1552. $$X_0 = x - \frac{1 + y'^2}{y''} y', \quad Y_0 = y + \frac{1 + y'^2}{y''}.$$

*The Evolute.*

When $y$, $y'$, and $y''$ are all explicit functions of $x$, the elimination of $x$ from the two equations 1552 gives an equation,

1553. $$F(X, Y) = 0,$$

which represents the locus of all the centres of curvature of the given curve. This locus is the *evolute* of the given curve.

The Evolute of the parabola $y^2 = 2dx$ is

1554. $Y^2 = \dfrac{8}{27} \dfrac{(X - d)^3}{d}$, a semi-cubic Parabola.

The Evolute of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is

1555. $$\left(\frac{aX}{a^2 - b^2}\right)^{\frac{2}{3}} + \left(\frac{bY}{a^2 - b^2}\right)^{\frac{2}{3}} = 1.$$

The Evolute of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is

1556. $$\left(\frac{aX}{a^2 + b^2}\right)^{\frac{2}{3}} - \left(\frac{bY}{a^2 + b^2}\right)^{\frac{2}{3}} = 1.$$

*Areas.*

The area included between the curve, $y = f(x)$, the axis of $x$ and the two ordinates $y_0$ and $y$ corresponding to the abscissas $x_0$ and $x$, is

1557. $$A = \int_{x_0}^{x} y\,dx = \int_{x_0}^{x} f(x)\,dx.$$

The area included between the curve and two radii vectores $r_0$ and $r$, corresponding to two values $\varphi_0$ and $\varphi$ of the polar angle, is

1558. $$A = \tfrac{1}{2}\int_{\phi_0}^{\phi} r^2 d\varphi.$$

In the parabola $y^2 = 4ax$, the area included between the vertex of the curve, the axis of $x$, the ordinate $y$, and the intercepted arc of the curve, is

1559. $$A = \int_0^x \sqrt{4ax}\,.dx = \tfrac{2}{3}xy.$$

In the circle $x^2 + y^2 = r^2$, the area included between the axis of $y$, the abscissa $x$, the ordinate $y$ and the intercepted arc of the curve, is

1560. $$A = \int_0^x \sqrt{r^2 - x^2}\,.dx = \frac{xy}{2} + \frac{r^2}{2}\sin^{-1}\frac{x}{r}.$$

The area of the whole circle, therefore, is $\pi r^2$.

In the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, the area included between the semi-axis minor, the abscissa $x$, the ordinate $y$, and the intercepted arc of the curve, is

1561. $$A = \int_0^x \frac{b}{a}\sqrt{a^2 - x^2}\,.dx = \frac{xy}{2} + \frac{ab}{2}\sin^{-1}\frac{x}{a}.$$

The area of the whole ellipse, therefore, is $\pi ab$.

In the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the area included between the vertex, the axis of $x$, the ordinate $y$, and the intercepted arc of the curve, is

1562. $$A = \int_a^{a+x} \frac{b}{a}\sqrt{x^2 - a^2}.dx = \frac{xy}{2} - \frac{ab}{2}\log_e\left(\frac{x}{a} + \frac{y}{b}\right),$$
$$= \frac{xy}{2} - \frac{ab}{2}\operatorname{Cosh}^{-1}\frac{x}{a}.$$

In the equilateral hyperbola $x^2 - y^2 = 1$, the last term of the right-hand member becomes $\frac{1}{2}\operatorname{Cosh}^{-1}x$, that is half the variable area $u$, of which the hyperbolic sine, cosine, tangent, etc., are defined as functions. See pp. 85–87.

*Lengths of Arcs.*

The length of the arc of a curve between two points corresponding to the two abscissas $x_0$ and $x$,

1563. $$s = \int_{x_0}^{x} \sqrt{dx^2 + dy^2} = \int_{x_0}^{x} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.dx.$$

Or, in polar coördinates,

1564. $$s = \int_{\phi_0}^{\phi} \sqrt{r^2 + \left(\frac{dr}{d\varphi}\right)^2}.d\varphi = \int_{r_0}^{r} \sqrt{1 + r^2\left(\frac{d\varphi}{dr}\right)^2}.dr.$$

*Envelopes.*

An envelope of a curve is the locus of the ultimate intersections of all curves of the series resulting from a continuous variation of the parameter of the curve.

1565. The equation of the envelope is obtained by eliminating the parameter $a$ between the equation of the curve

$$F(x, y, a) = 0 \quad\text{and}\quad D_aF(x, y, a) = 0.$$

The envelope of circles described on the central radii of the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ is

1566. $$(x^2 + y^2)^2 = a^2x^2 + b^2y^2.$$

*Pedal Curves.*

A *pedal curve* is the locus of the foot of a perpendicular drawn from the pole to a moving straight line which always touehes a given curve.

The pedal curve to a conic is

1567. $$(x^2 + y^2 + dx)^2 = a^2x^2 \pm b^2y^2,$$

wherein $a$ and $b$ are the semi-axes of the conic, and $d$ is the distance from the centre of the conic to the pole. The axes are rectangular with the origin at the pole, and the axis of $x$ coincides with the transverse axis of the conic.

*Special Cases.* (i) Let the pole be at the centre ($d = 0$) and the conic an equilateral hyperbola ($a^2 = b^2$). Then the equation becomes

1568. $(x^2 + y^2)^2 = a^2 (x^2 - y^2)$, a Lemniscate,
the polar equation of which is

$$r = a\sqrt{\cos 2\varphi}.$$

(ii) Let the pole be at a vertex of the conic ($d = -a$) and the conic a circle ($a^2 = b^2$).

Then the equation becomes

1569. $(x^2 + y^2 - ax)^2 = a^2 (x^2 + y^2)$, a Cardioid,
the polar equation of which is

$$r = a (1 + \cos \varphi).$$

(iii) Let the pole be at the distance $2a$ from the centre ($d = -2a$) and the conic a circle ($a^2 = b^2$). Then the equation becomes

1570. $(x^2 + y^2 - 2ax)^2 = a^2 (x^2 + y^2)$, a Limaçon,
the polar equation of which is

$$r = a (1 + 2 \cos \varphi).$$

*Trajectories.*

A trajectory is a curve which cuts according to a given law each curve of a series obtained by varying the parameter of $F (x, y, p) = 0$.

1571. The differential equation of the trajectory which cuts at a constant angle $\beta$ the series of curves $F(x, y, p)$ is obtained by eliminating $p$ from the equations

$$F(X, Y, p) = 0 \text{ and } \tan\beta = \frac{D_X F + D_Y F . D_X Y}{D_Y F - D_X F . D_X Y},$$

wherein $X$ and $Y$ are the running coördinates of the trajectory sought.

If the trajectory is to cut at right angles, $\tan\beta = \infty$, and the second equation becomes

$$D_Y F - D_X F . D_X Y = 0,$$

whence

1572. $\frac{dY}{dX} = \frac{D_Y F}{D_X F}$, the differential equation of the orthogonal trajectory. The integration of this equation gives a series of orthogonal trajectories.

*The Cycloid.*

A cycloid is the curve generated by a point in the circumference of a circle which rolls without sliding on a fixed straight line. Taking the fixed straight line, called the base of the cycloid, as the axis of $x$, and as origin one of the points where the generating point falls on the axis of $x$, the equations of the cycloid are

1573. $x = a\omega - a\sin\omega$ and $y = a - a\cos\omega$,

which by the elimination of $\omega$ give

1574. $$x = a\cos^{-1}\frac{a-y}{a} \pm \sqrt{(2a-y)\,y},$$

wherein $a$ denotes the radius of the rolling circle, and $\omega$ the measure of its turning around its centre.

Then

1575. $$dx = a(1-\cos\omega)\,d\omega,\quad dy = a\sin\omega d\omega$$

$$\tan\tau = \frac{dy}{dx} = \frac{\sin\omega}{1-\cos\omega} = \cot\tfrac{1}{2}\omega,$$

whence

1576. $$\tau = 90^\circ - \tfrac{1}{2}\omega.$$

Hence, if from the ends of the vertical diameter of the rolling circle in any position straight lines be drawn through the generating point, these lines will be tangent and normal to the cycloid.

1577. Normal $= 2a \sin \frac{1}{2}\omega = \sqrt{2ay}$.

Radius of curvature, $\rho = 4a \sin \frac{1}{2}\omega = 2\sqrt{2ay}$.

The evolute of a cycloid is composed of two halves of an equal cycloid so placed below the original cycloid that their summits are at the two ends of its base.

The area between the curve and its base from the origin to the ordinate $y$ is

1578. $$A = a^2 \left(\tfrac{3}{2}\omega - 2\sin\omega + \tfrac{1}{4}\sin 2\omega\right),$$
$$= \tfrac{3}{2}ax - \tfrac{1}{2}y\sqrt{(2a-y)\,y}.$$

The area between the whole curve and its base $= 3\pi a^2$.

The arc of the cycloid from the origin to the top of the ordinate $y$,

1579. $s = 4a\,(1 - \cos\frac{1}{2}\omega) = 4a - 2\sqrt{2a\,(2a-y)}$.

The whole arc $= 8a$.

A point rigidly connected with the rolling circle at a distance $p$ from its centre ($p$ being greater or less than the radius $a$) generates a curve of which the equations are

1580. $x = a\omega - p\sin\omega$ and $y = a - p\cos\omega$.

If $p > a$, the curve is a *prolate cycloid.*

If $p < a$, the curve is a *curtate cycloid.*

*Epicycloid and Hypocycloid.*

A point on the circumference of a circle which rolls, without sliding, on the circumference of a fixed circle generates an epicycloid or a hypocycloid according as the rolling circle is outside or inside the fixed circle.

Let $a$ be the radius of the fixed circle, $b$ that of the rolling circle; let A be the point on the fixed circle where the generating point P, at the start, touches the fixed circle; and let B be the point where the two circles touch at any subse-

quent time. The two arcs AB and PB are by definition of the curve equal; and, if $\alpha$ and $\beta$ be the angles which they subtend each in its own circle,

1581. $$a\alpha = \text{AB} = \text{PB} = b\beta,$$

whence $$\beta = \frac{a}{b}\alpha.$$

The equations, with the foregoing notation, and with the origin at the centre of the fixed circle, are for the

Epicycloid

1582. $$\begin{cases} x = (a+b)\cos\alpha - b\cos\left(\dfrac{a+b}{b}\alpha\right), \\ y = (a+b)\sin\alpha - b\sin\left(\dfrac{a+b}{b}\alpha\right). \end{cases}$$

Hypocycloid

1583. $$\begin{cases} x = (a-b)\cos\alpha + b\cos\left(\dfrac{a-b}{b}\alpha\right), \\ y = (a-b)\sin\alpha - b\sin\left(\dfrac{a-b}{b}\alpha\right). \end{cases}$$

In the Epicycloid,

1584. $$\tan\tau = \frac{dy}{dx} = \tan\left(\frac{a+2b}{2b}\alpha\right),\ \tau = \frac{a+2b}{2b}\alpha.$$

In the Hypocycloid,

1585. $$\tan\tau = \frac{dy}{dx} = -\tan\left(\frac{a-2b}{2b}\alpha\right),\ \tau = 180^\circ - \frac{a-2b}{2b}\alpha.$$

In the Epicycloid,

1586. $$\rho = \frac{4b\,(a+b)}{a+2b}\sin\left(\frac{a}{2b}\alpha\right).$$

In the Hypocycloid,

1587. $$\rho = \frac{4b\,(a-b)}{a-2b}\sin\left(\frac{a}{2b}\alpha\right).$$

The length of one loop is

1588. for the Epicycloid, $\dfrac{8\,(a+b)\,b}{a}$;

1589. for the Hypocycloid, $\dfrac{8\,(a-b)\,b}{a}$.

The area between one loop and the subtending arc of the fixed circle is

1590. for the Epicycloid, $\dfrac{\pi b^2 (3a + 2b)}{a}$,

1591. for the Hypocycloid, $\dfrac{\pi b^2 (3a - 2b)}{a}$.

1593. *Special Forms.* When $a$ and $b$ bear a commensurable ratio to each other, $\alpha$ may be eliminated from the equations of the epicycloid and hypocycloid and the result will be an algebraic equation in $x$ and $y$.

1594. (i) When $b = \infty$, $a$ remaining finite, the rolling circle becomes a straight line, any point of which generates the Involute of a Circle. See 1613.

1595. (ii) When $b = \frac{1}{2}a$, the hypocycloid becomes a straight line; and any point rigidly connected with the rolling circle, but not on its circumference, generates an ellipse.

1596. (iii) When $b = \frac{1}{4}a$, the hypocycloid takes the form of a four-pointed star, its equation being $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

1597. (iv) When $b = a$, the epicycloid becomes the Cardioid, the equation of which is, if the origin be moved to the point where the cardioid touches the fixed circle,

$$(y^2 + x^2 - 2ax)^2 = 4a^2 (y^2 + x^2)$$

or, in polar coördinates,

$$r = 2a (1 + \cos \varphi).$$

*The Epitrochoid and the Hypotrochoid.*

Any point rigidly connected with but not on the circumference of the rolling circle, and at a distance $p$ from its centre, generates a curve, of which the equations are

1598. $$\begin{cases} x = (a \pm b) \cos \alpha \mp p \cos\left(\dfrac{a \pm b}{b}\, \alpha\right), \\ y = (a \pm b) \sin \alpha - p \sin\left(\dfrac{a \pm b}{b}\, \alpha\right). \end{cases}$$

1599. When the rolling circle is outside the fixed circle the curve is an Epitrochoid, for which the above equations are to be read with the ***upper*** signs.

1600. When the rolling circle is inside the fixed circle the curve is a Hypotrochoid, for which the above equations are to be read with the *lower* signs.

1601. When $a = 2b$ the hypotrochoid becomes the ellipse

$$\frac{x^2}{(b+p)^2} + \frac{y^2}{(b-p)^2} = 1.$$

*The Catenary.* The curve of the hanging chain.

Equations,

1602.
$$\begin{cases} y = \dfrac{h}{2}\left(e^{\frac{x}{h}} + e^{-\frac{x}{h}}\right) = h \operatorname{Cosh} \dfrac{x}{h}. \\ x = h \log_e\left(\dfrac{y}{h} \pm \sqrt{\dfrac{y^2}{h^2} - 1}\right) = h \operatorname{Cosh}^{-1} \dfrac{y}{h}. \end{cases}$$

The axis of $x$ is horizontal, the axis of $y$ vertical, passing through the lowest point of the curve, and the origin at the distance $h$ below the lowest point of the curve. The slope of the tangent is

1603. $$\tan \tau = \frac{dy}{dx} = \frac{1}{2}\left(e^{\frac{x}{h}} - e^{-\frac{x}{h}}\right) = \operatorname{Sinh}\frac{x}{h} = \sqrt{\frac{y^2}{h^2} - 1}.$$

$$\cos \tau = \frac{h}{y}.$$

By making $\tau$ the independent variable, the following equations of the catenary are obtained:

1604.
$$\begin{cases} x = h \log_e \dfrac{1 + \sin \tau}{\cos \tau} = h \log_e \tan\left(\dfrac{\pi}{4} + \dfrac{\tau}{2}\right) = h \operatorname{Sinh}^{-1} (\tan \tau). \\ y = h \sec \tau. \end{cases}$$

1605. The radius of curvature, $\rho = \dfrac{y^2}{h} = h \sec^2 \tau.$

The area between the axis of $x$ and the curve, from $h$ the ordinate of the lowest point of the curve to any ordinate $y$,

1606.

$$A = \frac{h^2}{2}\left(e^{\frac{x}{h}} - e^{-\frac{x}{h}}\right) = h^2 \operatorname{Sinh}\frac{x}{h} = h^2 \tan\tau = h\sqrt{y^2 - h^2}.$$

The length of an arc measured from the lowest point of the curve to the top of any ordinate $y$,

1607. $$s = \frac{h}{2}\left(e^{\frac{x}{h}} - e^{-\frac{x}{h}}\right) = h \operatorname{Sinh}\frac{x}{h} = h\tan\tau = \sqrt{y^2 - h^2}.$$

Whence

1608. $$x = h\ \log_e\left[\frac{s}{h} + \sqrt{1 + \left(\frac{s}{h}\right)^2}\right] = h \operatorname{Sinh}^{-1}\frac{s}{h}.$$

The Involute of the Catenary is the Tractrix or the so-called Antifriction Curve. Its equation is

1609. $$\left(\frac{x}{h}\right)^2 = \left[\sqrt{1 - \frac{y^2}{h^2}} - \operatorname{Cosh}^{-1}\left(\pm\frac{h}{y}\right)\right]^2.$$

1610. The length of the tangent to this curve is constant $= h$.

The axis of $x$ is the asymptote to its four branches.

Its summit touches the lowest point of the catenary.

Another form of the equation of the Tractrix is

1611. $$x = h\log_e(h + \sqrt{h^2 - y^2}) - h\log_e y - \sqrt{h^2 - y^2}.$$

1612. The area included by the four branches of this curve is $\pi h^2$.

1613. *The Involute of a Circle.* Let there be given a fixed circle and in its plane a straight line moving so as always to touch, without sliding upon, its circumference; then any point in the moving tangent will generate the involute of the given circle. The end of a thread unwound from a fixed spool marks out such a curve.

Let A be the point where, at the start, the generating point is on the circumference of the fixed circle, T the point where the moving line in any other position touches the fixed circle, and P the generating point.

Then is the tangent PT always equal to the arc AT.

The radius of curvature, $\rho$ = PT. The measure of the arc AT is $r_0\theta$, wherein $r_0$ denotes the radius of the fixed circle.

The rectangular coördinates of P, the origin being at the centre of the fixed circle, are

1614. $$\begin{cases} x = r_0 (\cos\theta + \theta \sin\theta), \\ y = r_0 (\sin\theta - \theta\cos\theta). \end{cases}$$

The polar equation is

1615. $$\varphi = \sqrt{\frac{r^2}{r_0^2} - 1} - \tan^{-1}\sqrt{\frac{r^2}{r_0^2} - 1}.$$

1616. The length of the curve AP, $s = \dfrac{\rho^2}{2r_0} = \frac{1}{2}r_0\theta^2$.

1617. The area AOP $= \frac{1}{6}r_0^2\theta^3$.

1618. Since $\rho = r_0\theta$, the curve becomes a straight line when $\theta$ has increased to infinity.

### *Parabolic Curves.*

The family of Parabolic Curves has for its general equation

1619. $$y^m = ax^n.$$

Parabolic Spirals,

1620. $$r^m = a\varphi^n.$$

The Parabolic Spiral, a particular case,

1621. $$r^2 = a\varphi.$$

The Parabola of the $n^{\text{th}}$ degree,

1622. $$a^{n-1}y = x^n.$$

1623. The Cubic Parabola, $a^2y = x^3$.

1624. The Semi-cubic Parabola, $a^{\frac{1}{2}}y = x^{\frac{3}{2}}$.

1625. The Parabola referred to tangents at ends of the latus rectum,

$$x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}.$$

*The Spiral of Archimedes* is generated by a point moving at a uniform rate in a straight line while the straight line itself is turning at a uniform rate around a fixed point or pole.

1626. The polar equation is $r = a\varphi = \frac{r_0}{2\pi}\varphi$, wherein $r^0$ is the value of $r$ corresponding to $\varphi = 2\pi$.

1627. The polar subtangent $= \frac{r^2}{a}$.

1628. The polar subnormal $= a$, constant.

1629. The radius of curvature, $\rho = \frac{(a^2 + r^2)^{\frac{3}{2}}}{2a^2 + r^2}$.

1630. Length of the curve, $s = \frac{r_0}{4\pi}(\varphi\sqrt{1+\varphi^2} + \text{Sinh}^{-1}\varphi)$.

The family of *Hyperbolic Curves* has the general equation

1631. $$y^m x^n = a.$$

Hyperbolic Spirals,

1632. $$r^m \varphi^n = a.$$

1633. The Lituus, a particular case, $r^2\varphi = a$.

*The Hyperbolic Spiral.* Draw a diameter through a series of concentric circles, and on each circle beginning at this diameter measure off an arc of fixed length $a$, all in the same direction; the other ends of these arcs will be points in the hyperbolic spiral. Its equation is

1634. $$r\varphi = a.$$

This curve is also called the Reciprocal Spiral, since

$$r = \frac{a}{\varphi}.$$

1635. A straight line drawn parallel to the diameter used in the construction and at a distance $a$ therefrom is the asymptote of the spiral.

1636. The origin is an asymptotic point around which the spiral must make an infinite number of turns to reach it.

1637. The polar subtangent $= -a$, constant.

1638. The polar subnormal $= -\frac{r^2}{a}$.

Compare 1627, 1628.

1639. The radius of curvature, $\rho = \frac{r}{\cos^3 \theta}$, wherein $\theta$ is the angle between $r$ and the tangent.

*The Logarithmic Curve.*

1640. $$y = ae^{\frac{x}{n}} \text{ or } x = n \log_e \frac{y}{a},$$

wherein $n =$ the constant subtangent, and $a$ the intercept on the axis of $y$.

The most general form of the equation of this curve is

1641. $$y = ae^{\frac{x}{n}}\left(\cos \frac{2r\pi}{n} + i \sin \frac{2r\pi}{n}\right).$$

*The Logarithmic Spiral* is the locus of the equation

1642. $$r = ae^{m\phi}.$$

It is also called the Equiangular Spiral, since the tangent and radius vector form a constant angle.

The origin is an asymptotic point around which, for negative values of $\varphi$, the spiral must make an infinite number of turns to reach it.

The tangent and radius vector form a constant angle $\alpha$ with each other, and $\text{ctn}\, \alpha = m$. The value of $a$ is found by putting $\varphi = 0$.

1643. The polar normal $= r\sqrt{1 + m^2} = r \csc \alpha$.

1644. The polar subnormal $= r \,\text{ctn}\, \alpha = rm$.

1645. The radius of curvature $= r\sqrt{1 + m^2} = r$ cso $\alpha$.

The evolute is another spiral which will coincide with the original spiral by turning it through an angle

$$\frac{\pi}{2} - \frac{1}{m} \log_e m.$$

The area passed over by $r$ from $\varphi = -\infty$ (the pole) to any positive value of $\varphi$,

1646. $$A = \frac{r^2}{4m}.$$

1647. The length of the arc from the pole to any point,

$$s = r \sec \alpha.$$

*The Lemniscate.*

1648. $(x^2 + y^2)^2 = a^2 (x^2 - y^2), \ r = a\sqrt{\cos 2\varphi}.$

This curve is a particular case of the Cassinian Oval, see 1662.

*The Cissoid.*

1649. $$y^2 (2a - x) = x^3, \qquad r = 2\frac{a \sin^2 \varphi}{\cos \varphi}.$$

The area between the curve and its asymptote $= 3\pi a^2$.

*Descartes' Folium.*

1650. $$x^3 + y^3 = 3axy, \qquad r = \frac{3a \sin \varphi \cos \varphi}{\sin^3 \varphi + \cos^3 \varphi}.$$

The line $x + y + a = 0$ is an asymptote to this curve.

*Quadrifolium.*

1651. $$(x^2 + y^2)^3 = 4a^2x^2y^2, \qquad r = a \sin 2\varphi.$$

*The Witch of Agnesi*, or *The Versiera.*

1652. $$y = \frac{8a^3}{x^2 + 4a^2}.$$

*The Conchoid.*

If a radiant from a fixed point O cut a fixed straight line, the directrix, in R and a constant length RP $= b$ be measured from R either way along the radiant, the locus of P is a *conchoid.*

Let $a$ be the length of the perpendicular OB from O to the directrix. Then with B for origin of rectangular coördinates, the equation is

1653. $$x^2y^2 = (a + y)^2 (b^2 - y^2),$$

or, with O for pole,

$$r = a \sec \varphi \pm b.$$

*The Limaçon.* By changing the directrix of the conchoid into a fixed circle on OB $= a$ as diameter, the curve becomes a limaçon. With OB for initial line and axis of $x$, the equations of this curve are

1654. $r = a\cos\varphi \pm b, \quad (x^2 + y^2 - ax)^2 = b^2(x^2 + y^2),$
wherein $a =$ OB, $b =$ RP.

When $b > a$, O is a conjugate point.

When $b < a$, O is a node.

When $a = 2b$, this curve has been called the Trisectrix.

*The Folium,*

1655. $$y^2 = \frac{x^2(3a - x)}{3(a + x)}.$$

*The Logocyclic Curve,*

1656. $$y^2 = \frac{x(a - x)^2}{2a - x}, \qquad r = a(\sec\varphi \pm \tan\varphi).$$

*The Cubic Trisectrix,*

1657. $$y^2 = \frac{x^2(3a - x)}{a - x}, \qquad r = 2a\frac{\sin 3\varphi}{\sin 2\varphi},$$
$$\text{or} \quad r = 4a\cos\varphi - a\sec\varphi.$$

*The Quadratrix* is the locus of the intersection of the radius of a circle and an ordinate both of which move uniformly so that $x : a = \varphi : \frac{1}{2}\pi$. The equation is

1658. $$y = x\tan\left(\frac{a - x}{a}\cdot\frac{\pi}{2}\right).$$

*Cartesian Ovals.* The sum or difference of certain fixed multiples of the distances of a point P on the curve from two fixed points A and B, called the foci, is constant.

Let $r_1$ and $r_2$ be the focal radii and $c$ the distance between the foci.

The equations of the inner and outer ovals respectively are

1659. $mr_1 + lr_2 = nc$, 1660. $mr_1 - lr_2 = nc$,

wherein $n > m > l$.

*The Ovals of Cassini* are defined by the property that the product of the focal radii is constant

**1662.** $r_1r_2 = b^2$ or $[y^2 + (c+x)^2] \times [y^2 + (c-x)^2] = b^4$.

The polar equation is

**1663.** $$r^4 - 2c^2r^2 \cos 2\varphi = b^4 - c^4,$$

the pole being half way between the foci, whose distance apart $= 2c$.

The equation

**1664.** $$r = a (\sec \varphi + n \cos \varphi)$$

represents a family of curves to which belong the Cissoid and the Cubic Trisectrix.

*Miscellaneous Polar Equations.*

**1665.** $r = a \sin \varphi$ (a circle). **1666.** $r = a \sin \frac{1}{2}\varphi$.

**1667.** $r = a \sin 2\varphi$. **1668.** $r \cos \varphi = a \cos 2\varphi$.

**1669.** $r = a \sin 3\varphi$. **1670.** $r = a \cos \varphi + b$.

**1671.** $r = a \sin n\varphi$ ($n$ loops when $n$ is odd, $2n$ loops when $n$ is even.

**1672.** $r = \sin^3 \frac{1}{3}\varphi$. **1673.** $= a + b \csc \varphi$.

**1674.** $r = a \tan \varphi$. **1675.** $r = a \sec^2 \frac{1}{2} \varphi$ (a parabola).

**1676.** $r = a (\sec 2\varphi + \tan 2\varphi)$.

*Miscellaneous Rectangular Equations.*

**1677.** $y^2 = \dfrac{x^3 - a^3}{x - 2a}$. **1678.** $y^2 = x \log (1 + x)$.

**1679.** $y^2 = \dfrac{x^3}{x - a}$. **1680.** $a^2y^2 = a^2x^2 - x^4$.

**1681.** $y^2 = x^2 \dfrac{a + x}{a - x}$. **1682.** $ay^2 = (x - a)^2 (x - b)$.

**1683.** $y^2 = \pm \dfrac{x^4}{a^2 - x^2}$. **1684.** $ay^2 = x^3 \pm bx^2$.

1685. $y^2 = x(x+a)^2.$ 1686. $x^4 - a^2xy + b^2y^2 = 0.$

1687. $y^2 = \dfrac{x^4}{a^2} - \dfrac{x^6}{a^4}.$ 1688. $a^3y^2 - 2abx^2y + x^5 = 0.$

1689. $y^3 \pm x^3 = 2ax^2.$

1690. $x^3 + 2x^2 + 2xy - y^2 + 5x - 2y = 0.$

1691. $ay^3 + axy^2 - x^4 = 0.$ 1692. $(y^2 - x^2)^2 = x^5.$

1693. $ay^3 - 2ax^2y - x^4 = 0.$ 1694. $y = \dfrac{x^3}{a^2 + x^2}.$

1695. $ax^3 + by^3 - c = 0.$ 1696. $a^3y - a^2x^2 + x^4 = 0.$

1697. $x^3 + y^3 = a^3.$ 1698. $a^2y - 3bx^2 + x^3 = 0.$

1699. $\left(\dfrac{x}{a}\right)^3 + \left(\dfrac{y}{b}\right)^3 = 1.$ 1700. $y = x^4 - 4x^3 - 18x^2.$

1701. $\left(\dfrac{x}{a}\right)^2 + \left(\dfrac{y}{b}\right)^{\frac{2}{3}} = 1.$ 1702. $y = \log \sec x.$

1703. $x^2(x+y) = a^2(x-y).$

1704. $\left(\dfrac{x}{a}\right)^{2n+1} + \left(\dfrac{y}{b}\right)^{2n+1} = 1.$ 1705. $\left(\dfrac{x}{a}\right)^{2n} + \left(\dfrac{y}{b}\right)^{2n} = 1.$

1706. $y(y^2 - 1) = (x^2 - \frac{9}{4})(x^2 - \frac{1}{4}).$

1707. $y^4 - 96a^2y^2 + 100a^2x^2 - x^4 = 0.$

1708. $y^4 + x^4 = a^2(y^2 - x^2).$

1709. $x^2y^2 - 2axy - b^2(y^2 + b^2) + 2a^2b^2 = 0.$

1710. $x^5 - 2a^3xy + y^5 = 0.$

1711. *The Probability Curve,* $y = e^{-x^2}.$

*The Point, the Straight Line, and the Plane in Space.*

The coördinates of a point $(x, y, z)$ dividing the distance between two points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ in a given ratio $l : m$ are

1802. $$x = \frac{lx_2 + mx_1}{l + m}, \quad y = \frac{ly_2 + my_1}{l + m}, \quad z = \frac{lz_2 + mz_1}{l + m}.$$

The point of external division is given by changing $m$ to $-m$.

The distance between two points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$, is

1803. $$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

If the angles which the line $d$ makes with the axes of $x$, $y$, $z$, be $\alpha$, $\beta$, $\gamma$, respectively, then

1804. $$\cos\alpha = \frac{x_1 - x_2}{d}, \quad \cos\beta = \frac{y_1 - y_2}{d}, \quad \cos\gamma = \frac{z_1 - z_2}{d}.$$

The distance of any point $(x, y, z)$ from the origin is

1805. $$r = \sqrt{x^2 + y^2 + z^2}.$$

A much used system of polar coördinates consists of the radius vector $r$ and the three angles $\alpha$, $\beta$, $\gamma$ which $r$ makes with the axes of $x$, $y$, and $z$ respectively. Then for any point $(x, y, z)$ or $(r, \alpha, \beta, \gamma)$

1806. $$\begin{cases} r^2 = x^2 + y^2 + z^2, \\ x = r\cos\alpha, \quad y = r\cos\beta, \quad z = r\cos\gamma. \end{cases}$$

These cosines, usually called the *direction cosines*, are connected by the relation

1807. $$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

The angle $\theta$ between two lines whose direction cosines are known is given by the equation,

1808. $$\cos\theta = \cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2,$$

or by the equation

1809. $$\begin{aligned} \sin^2\theta = {} & (\cos\beta_1\cos\gamma_2 - \cos\beta_2\cos\gamma_1)^2 \\ & + (\cos\gamma_1\cos\alpha_2 - \cos\gamma_2\cos\alpha_1)^2 \\ & + (\cos\alpha_1\cos\beta_2 - \cos\alpha_2\cos\beta_1)^2. \end{aligned}$$

The condition that two lines be at right angles to each other is

1810. $$\cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2 = 0.$$

The direction cosines of a line perpendicular to two given lines and therefore to their plane are given by the equations

1811. $$\begin{cases} \sin\theta\cos\alpha = \cos\beta_1\cos\gamma_2 - \cos\beta_2\cos\gamma_1 \\ \sin\theta\cos\beta = \cos\gamma_1\cos\alpha_2 - \cos\gamma_2\cos\alpha_1 \\ \sin\theta\cos\gamma = \cos\alpha_1\cos\beta_2 - \cos\alpha_2\cos\beta_1 \end{cases}$$

wherein $\alpha$, $\beta$, $\gamma$, are the direction angles of the perpendicular to be found, $\alpha_1, \beta_1, \gamma_1$ and $\alpha_2, \beta_2, \gamma_2$, the direction angles of the given lines, and $\theta$ the angle between them. If $\theta = 90°$, $\sin\theta = 1$; and the above equations then give the direction cosines of one of three mutually perpendicular lines in terms of the direction cosines of the other two.

*Transformation of Coördinates.*

*Parallel Transformation.* The coördinates in the old system being $x, y, z$; those in the new $x', y', z'$; and the coördinates of the new origin relatively to the old axes being $x_0, y_0, z_0$; the transformation is made by substituting for $x, y, z$ their equivalents.

1812. $$x = x_0 + x',\ y = y_0 + y',\ z = z_0 + z'.$$

*Turning rectangular axes around the origin still keeping them rectangular.* Let the angles made by the new axes of $x, y, z$ with the old axes be $\alpha_1, \beta_1, \gamma_1$; $\alpha_2, \beta_2, \gamma_2$; $\alpha_3, \beta_3, \gamma_3$, respectively; and for shortness put $\cos\alpha_1 = a_1$, $\cos\beta_1 = b_1$, etc.

1813.
$$\begin{cases} x = a_1x' + a_2y' + a_3z' \\ y = b_1x' + b_2y' + b_3z' \\ z = c_1x' + c_2y' + c_3z'. \end{cases}$$

1814.
$$\begin{cases} x' = a_1x + b_1y + c_1z \\ y' = a_2x + b_2y + c_2z \\ z' = a_3x + b_3y + c_3z. \end{cases}$$

The following relations between these nine direction cosines hold good,

1815.
$$\begin{cases} a_1^2 + a_2^2 + a_3^2 = 1 \\ b_1^2 + b_2^2 + b_3^2 = 1 \\ c_1^2 + c_2^2 + c_3^2 = 1. \end{cases}$$

1816.
$$\begin{cases} a_1^2 + b_1^2 + c_1^2 = 1 \\ a_2^2 + b_2^2 + c_2^2 = 1 \\ a_3^2 + b_3^2 + c_3^2 = 1. \end{cases}$$

1817.
$$\begin{cases} a_1b_1 + a_2b_2 + a_3b_3 = 0 \\ b_1c_1 + b_2c_2 + b_3c_3 = 0 \\ c_1a_1 + c_2a_2 + c_3a_3 = 0. \end{cases}$$

1818.
$$\begin{cases} a_1a_2 + b_1b_2 + c_1c_2 = 0 \\ a_2a_3 + b_2b_3 + c_2c_3 = 0 \\ a_3a_1 + b_3b_1 + c_3c_1 = 0. \end{cases}$$

**1819.**

$$\begin{cases} a_1 = b_2c_3 - b_3c_2 \\ b_1 = c_2a_3 - c_3a_2 \\ c_1 = a_2b_3 - a_3b_2. \end{cases}$$

**1820.**

$$\begin{cases} a_2 = c_1b_3 - c_3b_1 \\ b_2 = a_1c_3 - a_3c_1 \\ c_2 = b_1a_3 - b_3a_1. \end{cases}$$

**1821.**

$$\begin{cases} a_3 = b_1c_2 - b_2c_1 \\ b_3 = c_1a_2 - c_2a_1 \\ c_3 = a_1b_2 - a_2b_1. \end{cases}$$

Another system of polar coördinates, used in Astronomy, consists of

$r =$ the radius vector,

$\theta =$ the angle which $r$ makes with its projection on the plane of $xy$,

$\varphi =$ the angle which this projection of $r$ makes with the axis of $x$.

**1822.** These polar coördinates are related to rectangular coördinates as follows:

$$x = r\cos\theta\cos\varphi, \qquad y = r\cos\theta\sin\varphi, \qquad z = r\sin\theta.$$
$$r^2 = x^2 + y^2 + z^2.$$

*The general form of the equation of a plane* is

**1823.** $$Ax + By + Cz + D = 0.$$

If $A = 0$, the plane is parallel to the axis of $x$;
$B = 0$, the plane is parallel to the axis of $y$;
$C = 0$, the plane is parallel to the axis of $z$.

If $D = 0$, the plane passes through the origin.

If $A = 0$ and $B = 0$, the plane is parallel to the $xy$-plane.
$B = 0$ and $C = 0$, the plane is parallel to the $yz$-plane.
$C = 0$ and $A = 0$, the plane is parallel to the $zx$-plane.

If the plane makes intercepts $a, b, c$ on the axes of $x, y, z$ respectively, the equation is

**1824.** $$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

If a perpendicular from the origin upon a plane be $p$ in length and make the angles $\alpha, \beta, \gamma$ with the axes of $x$, $y$, $z$, the equation of the plane takes the *normal* form

1825. $$x \cos\alpha + y \cos\beta + z \cos\gamma = p.$$

An equation of the form $Ax + By + Cz + D = 0$ is reduced to the normal form by putting

1826.
$$\left\{\begin{array}{ll} \pm\dfrac{A}{\sqrt{A^2 + B_2 + C^2}} = \cos\alpha, & \pm\dfrac{C}{\sqrt{A^2 + B^2 + C^2}} = \cos\gamma, \\ \pm\dfrac{B}{\sqrt{A^2 + B^2 + C^2}} = \cos\beta, & \pm\dfrac{-D}{\sqrt{A^2 + B^2 + C^2}} = p. \end{array}\right.$$

N.B. The sign of the radical is to be so chosen as to give $p$ a positive value.

The angle $\theta$ between two planes

$$\begin{aligned} Ax + By + Cz + D &= 0 \\ A'x + B'y + C'z + D' &= 0 \end{aligned}$$

is given by either of the equations

1827. $$\cos\theta = \frac{AA' + BB' + CC'}{\sqrt{A^2 + B^2 + C^2}\sqrt{A'^2 + B'^2 + C'^2}},$$

1828. $$\sin^2\theta = \frac{(BC' - B'C)^2 + (CA' - C'A)^2 + (AB' - A'B)^2}{(A^2 + B^2 + C^2)(A'^2 + B'^2 + C'^2)}.$$

The condition that the planes should be perpendicular to each other is

1829. $$AA' + BB' + CC' = 0;$$

and the condition that they should be parallel is

1830. $$\frac{A}{A'} = \frac{B}{B'} = \frac{C}{C'}.$$

The length of a perpendicular from a given point $(x_1, y_1, z_1)$ to a given plane, $x \cos \alpha + y \cos \beta + z \cos \gamma = p$, is

1831. $$l = \pm (x_1 \cos \alpha + y_1 \cos \beta + z_1 \cos \gamma - p),$$

wherein the positive sign is to be taken when the origin and the point $(x_1, y_1, z_1)$ are on opposite sides of the plane, and the negative sign when they are on the same side of the plane.

If the equation of the given plane is of the form

$$Ax + By + Cz + D = 0,$$

the length of the perpendicular is given in the form

1832. $$l = \frac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}}.$$

All points for which the numerator of this fraction has the same sign as $D$ are on the same side of the plane as the origin.

The equation

1833. $$(Ax + By + Cz + D) + k\,(A'x + B'y + C'z + D') = 0,$$

wherein $k$ is arbitrary, represents all planes passing through the intersection of the two planes

$$Ax + By + Cz + D = 0$$
$$A'x + B'y + C'z + D' = 0.$$

1834. If $k$ be given alternately one of the two values

$$\pm \frac{\sqrt{A^2 + B^2 + C^2}}{\sqrt{A'^2 + B'^2 + C'^2}},$$

the resulting equations will represent the two planes perpendicular to each other which bisect the supplemental angles between the two given planes.

Or, if the equations of the given planes are in the normal form 1825, the equations of the two bisecting planes are

1835. $$(x \cos \alpha + y \cos \beta + z \cos \gamma - p) \pm (x \cos \alpha' + y \cos \beta' + z \cos \gamma' - p') = 0.$$

The condition that four planes should pass through one and the same point is that the determinant formed by eliminating $x$, $y$, $z$ from their four equations should vanish; that is

1836. $$\begin{vmatrix} A & B & C & D \\ A' & B' & C' & D' \\ A'' & B'' & C'' & D'' \\ A''' & B''' & C''' & D''' \end{vmatrix} = 0.$$

1837. This is also the condition that two straight lines in space intersect, each line being represented by two of these equations.

1838. Six times the volume of the tetrahedron whose vertices are $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$, $(x_3, y_3, z_3)$, $(x_4, y_4, z_4)$

is the determinant $$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

The condition that four points lie all in one plane is that this determinant vanish.

Hence the equation of a plane passing through three given points may be written

1839. $$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

*A straight line in space* is determined by two equations of the general form

1840. $$Ax + By + Cz + D = 0$$
$$A'x + B'y + C'z + D' = 0.$$

Each of these equations represents a plane, and the straight line represented is the intersection of these two planes.

By eliminating $y$ and $x$ alternately between these equations, they assume the explicit form commonly used,

1841. $$x = mz + a, \qquad y = nz + b.$$

The first is the equation of the *projection* of the straight line on the plane of $xz$, and the second that of its projection on the plane of $yz$. Or, regarding these equations as representing planes, the first represents a plane parallel to the axis of $y$, the second a plane parallel to the axis of $x$, the straight line they represent being the intersection of these two planes.

If the straight line passes through a given point $(x_1, y_1, z_1)$ and forms angles $\alpha, \beta, \gamma$, with the axes of $x$, $y$, and $z$, its equations are

1842. $$\frac{x - x_1}{\cos\alpha} = \frac{y - y_1}{\cos\beta} = \frac{z - z_1}{\cos\gamma}.$$

If the straight line passes through two given points, $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$, its equations are

1843. $$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} = \frac{z - z_1}{z_1 - z_2}.$$

The equations of a straight line passing through the point $(x_1, y_1, z_1)$ perpendicular to the plane

$$Ax + By + Cz + D = 0 \text{ are}$$

1844. $$\frac{x - x_1}{A} = \frac{y - y_1}{B} = \frac{z - z_1}{C}.$$

1845. If the equations of a straight line are brought into the form

$$\frac{x - x_1}{A} = \frac{y - y_1}{B} = \frac{z - z_1}{C},$$

the direction cosines of the line are $A$, $B$, and $C$ each divided by $\sqrt{A^2 + B^2 + C^2}$.

The angle $\theta$ between two lines

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n} \text{ and } \frac{x-a}{l'}=\frac{y-b}{m'}=\frac{z-c}{n'}$$

is given by the equation

1846. $$\cos\theta=\frac{ll'+mm'+nn'}{\sqrt{l^2+m^2+n^2}\sqrt{l'^2+m'^2+n'^2}}.$$

The lines are perpendicular to each other if

1847. $$ll'+mm'+nn'=0.$$

The angle $\theta$ between the plane $Ax+By+Cz+D=0$ and the line $\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}$ is given by the equation

1848. $$\sin\theta=\frac{Al+Bm+Cn}{\sqrt{l^2+m^2+n^2}\sqrt{A^2+B^2+C^2}}.$$

The line is parallel to the plane if

1849. $$Al+Bm+Cn=0.$$

1850. The conditions that the straight line

$$x=mz+a, \quad y=nz+b$$

lie wholly in the plane

$$Ax+By+Cz+D=0$$

are

$$Aa+Bb+D=0$$
$$Am+Bn+C=0.$$

If there be two straight lines given by the equations

$$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1} \text{ and } \frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c^2}$$

wherein $a_1$, $b_1$, $c_1$, $a_2$, $b_2$, $c_2$, are the direction cosines of the two lines, the equations of planes passing through each line parallel to the other are

1851. $$(x-x_1)(b_1c_2-b_2c_1)+(y-y_1)(c_1a_2-c_2a_1) + (z-z_1)(a_1b_2-a_2b_1)=0$$

$$(x-x_2)(b_1c_2-b_2c_1)+(y-y_2)(c_1a_2-c_2a_1) + (z-z_2)(a_1b_2-a_2b_1)=0.$$

These planes are parallel; and the distance between them, which is also the shortest distance between the given lines, is equal to the difference of their absolute terms divided by the square root of the sum of the squares of the coefficients of $x$, $y$, $z$. Thus

1852. $$d = \frac{(x_1 - x_2)(b_1c_2 - b_2c_1) + (y_1 - y_2)(c_1a_2 - c_2a_1) + (z_1 - z_2)(a_1b_2 - a_2b_1)}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}.$$

The denominator of this expression is equal to $\sin\theta$, where $\theta$ is the angle which the directions of the two given lines make with each other. See **1809.**

*Quadrics.*

*The general equation of the second degree in three variables* is

1853. $$ax^2 + by^2 + cz^2 + d + 2fyz + 2gzx + 2hxy + 2lx + 2my + 2nz = 0.$$

The Discriminant is

1854. $$\Delta = \begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & d \end{vmatrix} = abcd + 2afmn + 2bgnl + 2chlm + 2dfgh - bcl^2 - cam^2 - abn^2 - adf^2 - bdg^2 - cdh^2 + f^2l^2 + g^2m^2 + h^2n^2 - 2ghmn - 2hfnl - 2fglm.$$

The derivative relative to $d$ of the Discriminant is

1855. $$D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2.$$

1856. If $D$ be *not* $= 0$, the surface represented is a *central* surface, either an ellipsoid, a hyperboloid (of one sheet or of two), or a cone.

1857. If $D = 0$, the surface represented is some form of cylinder (elliptic, hyperbolic, or parabolic) or some form of paraboloid.

1858. If $\Delta = 0$, the surface represented is a cone.

For further discrimination, the *discriminating cubic equation*, **1862**, should be used.

*Transformation of the General Equation.*

If $D$ be *not* 0, and therefore the surface be *central*, the general equation can be transformed to parallel axes through the centre.

**1859.** The coördinates $x_0, y_0, z_0$ of the centre are found by solving the equations

$$\begin{aligned} ax_0 + hy_0 + gz_0 + l &= 0, \\ hx_0 + by_0 + fz_0 + m &= 0, \\ gx_0 + fy_0 + cz_0 + n &= 0. \end{aligned}$$

These equations express the condition that the new $l$, $m$, $n$ in the transformed equation should each be equal to 0.

Thus the equation is freed of terms of the first degree and becomes

**1860.** $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + d' = 0,$

wherein $$d' = \frac{\Delta}{D}.$$

**1861.** Further, the terms in $yz$, $zx$, and $xy$ can be made to disappear by turning the axes on the centre so as to make the new $f$, $g$, and $h$ vanish. This can be done by virtue of the relations that exist between the old and new coefficients when an equation is transformed from one set of rectangular axes to another with the same origin.

These relations are

$$\begin{gathered} a + b + c = a' + b' + c', \\ bc + ca + ab - f^2 - g^2 - h^2 = b'c' + c'a' + a'b' - f'^2 - g'^2 - h'^2, \\ abc + 2fgh - af^2 - bg^2 - ch^2 = a'b'c' + 2f'g'h' - a'f'^2 - b'g'^2 - c'h'^2. \end{gathered}$$

By making $f'$, $g'$, and $h'$ each $= 0$, these equations enable us to form a cubic equation whose roots are $a'$, $b'$, and $c'$ (for their sum, their product and the sum of their products taken two by two have all become known).

This cubic equation is known as the *discriminating cubic*, and it is written

1862. $$k^3 - (a+b+c)k^2 + (bc+ca+ab-f^2-g^2-h^2)k - (abc+2fgh-af^2-bg^2-ch^2) = 0.$$

The three values of $k$, which are all real, found by solving this cubic, are the $a'$, $b'$, and $c'$ sought.

The general equation now becomes

1863. $$a'x^2 + b'y^2 + c'z^2 + d' = 0.$$

This will assume different forms according to the nature of the roots of the discriminating cubic.

(i) Let all three roots be positive. The surface then makes a real intercept on each axis, is a closed surface, and if $a$, $b$, and $c$ denote these intercepts, the equation may be written in the form

1864. $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad \textit{An Ellipsoid.}$$

If $a = b = c$, the surface is that of a Sphere.

If $a = b > c$, the surface is that of an Oblate Spheroid.

If $a = b < c$, the surface is that of a Prolate Spheroid.

(ii) Let one root be negative. Then

1865. $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \quad \textit{A Hyperboloid of one sheet.}$$

If $a = b$, it is a surface of revolution.

(iii) Let two of the roots be negative. Then

1866. $$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \quad \textit{A Hyperboloid of two sheets.}$$

If $b = c$, it is a surface of revolution.

(iv) Let all three roots be negative. Then

1867. $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = -1, \quad \textit{No real locus.}$$

(v) When $d' = \dfrac{\Delta}{D}$ vanishes in consequence of $\Delta = 0$,

1868. $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0$, *An infinitely small ellipsoid at the origin.*

1869. $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$, *A Cone, asymptotic to the corresponding hyperboloid.*

If $D=0$, the surface represented by the general equation is *not* central, and transformation to parallel axes through the centre is impossible; but the axes may be turned, without changing the origin, so as to make the terms in $yz$, $zx$, and $xy$ disappear.

In doing this, observe that the last term of the discriminating cubic $=D=0$, and therefore one of the roots $a'$, $b'$, $c'$ is equal to 0. Say $c'=0$. Thus the equation becomes

1870. $a'x^2 \pm b'y^2 + 2l'x + 2m'y + 2n'z + d = 0.$

Transformation to parallel axes through a new origin can make the coefficients of $x$ and $y$ vanish but not that of $z$; and the equation becomes

1871. $a'x^2 \pm b'y^2 + 2n'z + d' = 0,$

which represents the following surfaces.

(i) If $n'=0$,

1872. $a'x^2 + b'y^2 + d' = 0$, *Elliptic Cylinder.*

If $a'=b'$, the cylinder is circular.

1873. $a'x^2 - b'y^2 + d' = 0$, *Hyperbolic Cylinder.*

The axis of these cylinders is the coördinate axis of $z$.

If also $d'=0$,

1874. $a'x^2 - b'y^2 = 0$, *A pair of planes*

intersecting in the axis of $z$ and asymptotic to the hyperbolic cylinder $a'x^2 - b'y^2 + d' = 0$.

(ii) If $n'$ be not $=0$, a change of origin will make $d'$ disappear and the equation will take one or the other of the two following forms

1875. $a'x^2 + b'y^2 + 2n'z = 0$, *An Elliptic Paraboloid.*

If $a' = b'$, the paraboloid is one of revolution.

1876. $a'x^2 - b'y^2 + 2n'z = 0$, *A Hyperbolic Paraboloid.*

(iii) If also $b' = 0$, that is, if two roots of the discriminating cubic are 0, the equation takes the form

$$a'x^2 + 2m'y + 2n'z + d = 0,$$

which, by a change of axes in the plane $yz$, becomes

1877. $a'x^2 + 2m'y + d = 0$, *A Parabolic Cylinder.*

(iv) If $c' = 0$, $b' = 0$, $m' = 0$, and $n' = 0$, the equation becomes

1878. $a'x^2 + d = 0$, *A pair of parallel planes.*

Every homogeneous equation of the second degree in three variables,

1879. $Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fzx = 0$,

represents a cone.

Every equation of the form

1880. $x^2 + y^2 + z^2 + Ax + By + Cz + D = 0$

represents a sphere.

If $r$ be the radius and $a$, $b$, $c$ the coördinates of the centre of the sphere, its equation takes the form

1881. $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$.

And if the origin be at the centre, its equation takes the form

1882. $x^2 + y^2 + z^2 = r^2$.

*Curved Surfaces.*

The general equation of a curved surface is

$$F(x, y, z) = 0,$$

which, by solving for $z$, becomes

1883. $z = f(x, y)$.

Equation of the plane tangent at the point $(x, y, z)$,

1884. $(X - x) D_xF + (Y - y) D_yF + (Z - z) D_zF = 0$.

Equations of the normal at the point $(x, y, z)$,

1885. $$\frac{X-x}{D_xF} = \frac{Y-y}{D_yF} = \frac{Z-z}{D_zF}.$$

The angles $\alpha$, $\beta$, $\gamma$, which the normal makes with the coördinate axes are determined by

1886. $$\cos\alpha = \frac{D_xF}{R}, \cos\beta = \frac{D_yF}{R}, \cos\gamma = \frac{D_zF}{R},$$

wherein

$$R = \sqrt{(D_xF)^2 + (D_yF)^2 + (D_zF)^2}.$$

Applying the foregoing formulas to the general equation of quadrics **1853**, we have

$$F(x, y, z) = ax^2 + by^2 + cz^2 + d + 2fyz + 2gzx + 2hxy + 2lx + 2my + 2nz = 0,$$

which for shortness shall be written $u = 0$.

Then,

1887. $$\begin{cases} D_xF = D_xu = 2(ax + hy + gz + l) \\ D_yF = D_yu = 2(hx + by + fz + m) \\ D_zF = D_zu = 2(gx + fy + cz + n), \end{cases}$$

and the equation of the plane tangent at the point $(x, y, z)$ becomes

1888. $$\left.\begin{aligned} &(X-x)(ax + hy + gz + l) \\ +&(Y-y)(hx + by + fz + m) \\ +&(Z-z)(gx + fy + cz + n) \end{aligned}\right\} = 0,$$

or, writing at full length and reducing,

1889. $$\left.\begin{aligned} &X(ax + hy + gz + l) \\ +&Y(hx + by + fz + m) \\ +&Z(gx + fy + cz + n) \\ +&lx + my + nz + d \end{aligned}\right\} = 0.$$

The equations of the normal at the point $(x, y, z)$ become

1890. $$\frac{X-x}{ax+hy+gz+l}=\frac{Y-y}{hx+by+fz+m}=\frac{Z-z.}{gx+fy+cz+n},$$

and the direction cosines of the normal are

1891. $$\frac{ax+hy+gz+l}{R}, \frac{hx+by+fz+m}{R}, \frac{gx+fy+cz+n}{R},$$

wherein $R =$

$$\sqrt{(ax+hy+gz+l)^2+(hx+by+fz+m)^2+(gx+fy+cz+n)^2}.$$

The equation

1892. $$\frac{xx'}{a^2}+\frac{yy'}{b^2}+\frac{zz'}{c^2}=1,$$

represents the *polar plane* of the point $(x', y', z')$ relatively to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1;$$

or, if the point $(x', y', z')$ be on the surface, it represents the *tangent plane* at that point. By changing the sign of $c^2$ or of both $b^2$ and $c^2$ the same statement and formulas apply to the hyperboloid of one sheet or of two.

The length of a perpendicular, $p$, from the origin upon the tangent plane is given by

1893. $$\frac{1}{p^2}=\frac{x'^2}{a^4}+\frac{y'^2}{b^4}+\frac{z'^2}{c^4}.$$

The angles $\alpha$, $\beta$, $\gamma$, which this perpendicular makes with the coördinate axes are given by

1894. $$\cos\alpha=\frac{px'}{a^2}, \cos\beta=\frac{py'}{b^2}, \cos\gamma=\frac{pz'}{c^2}.$$

Whence,

1895. $$p^2=a^2\cos^2\alpha+b^2\cos^2\beta+c^2\cos^2\gamma.$$

The condition that the plane

$$Ax + By + Cz + D = 0$$

should touch the surface

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = 1$$

is that

1896. $$A^2a^2 + B^2b^2 + C^2c^2 = D^2.$$

The equations of the normal at the point $(x', y', z')$ are

1897. $$\frac{a^2}{x'}(x - x') = \frac{b^2}{y'}(y - y') = \frac{c^2}{z'}(z - z').$$

The equation

1898. $$\frac{xx'}{a^2} \pm \frac{yy'}{b^2} \pm \frac{zz'}{c^2} = 0$$

represents the *diametral plane* which is conjugate to the diameter drawn from the point $(x', y', z')$ on the surface, and is parallel to the plane tangent at that point.

If two diameters are conjugate to each other their direction cosines are thus related,

1899. $$\frac{\cos\alpha\cos\alpha'}{a^2} + \frac{\cos\beta\cos\beta'}{b^2} + \frac{\cos\gamma\cos\gamma'}{c^2} = 0.$$

### *Curves of Double Curvature.*

A curve of double curvature is represented by two equations

1900. $$F_1(x, y, z) = 0, \quad F_2(x, y, z) = 0.$$

The curve is the intersection of the two surfaces which these two equations taken separately represent.

This curve is also represented by the equations of its projections on two of the coördinate planes, thus,

1901. $$y = f_1(x) \text{ and } z = f_2(x),$$

which are the equations of two cylindrical surfaces the elements of which are parallel to the axes of $z$ and $y$ respectively.

The angles $\alpha$, $\beta$, $\gamma$ which the *tangent* to a curve of double curvature makes with the axes of coördinates are determined by the equations

1902. $$\cos\alpha = \frac{dx}{ds},\ \cos\beta = \frac{dy}{ds},\ \cos\gamma = \frac{dz}{ds},$$

wherein

1903. $$ds = \sqrt{dx^2 + dy^2 + dz^2} = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2}.$$

The equations of the *tangent* at the point $(x, y, z)$ are

1904. $$\frac{X-x}{dx} = \frac{Y-y}{dy} = \frac{Z-z}{dz},$$

The *normal plane* is the plane containing all the infinite number of normal lines that can be drawn at any point of the curve. It is perpendicular to the tangent, and its equation, when it passes through the point $(x, y, z)$ is

1905. $$(X-x)\,dx + (Y-y)\,dy + (Z-z)\,dz = 0.$$

The *osculating plane* is the plane which passes through three consecutive points of the curve and therefore contains two consecutive tangents. Its equation, when it touches the curve at the point $(x, y, z)$ is

1906. $$A(X-x) + B(Y-y) + C(Z-z) = 0,$$

wherein, for shortness,

$$\begin{aligned} A &= dyd^2z - dzd^2y \\ B &= dzd^2x - dxd^2z \\ C &= dxd^2y - dyd^2x. \end{aligned}$$

The angles $\lambda$, $\mu$, $\nu$ which the normal to the osculating plane, the so-called *binormal* makes with the coördinate axes are determined by the equations

1907. $$\cos\lambda = \frac{A}{R},\ \cos\mu = \frac{B}{R},\ \cos\nu = \frac{C}{R},$$

1908. wherein $R = \sqrt{A^2 + B^2 + C^2}$

$$= ds\sqrt{(d^2x)^2 + (d^2y)^2 + (d^2z)^2 - (d^2s)^2},$$

$$= ds^2\sqrt{\left(d\,\frac{dx}{ds}\right)^2 + \left(d\,\frac{dy}{ds}\right)^2 + \left(d\,\frac{dz}{ds}\right)^2}.$$

Let the angle between two consecutive tangents be denoted by $d\tau$; then

1909. $$d\tau = \frac{R}{ds^2},$$

and the radius of *curvature in the osculating plane*, or the so-called *first curvature* of the curve, is

1910. $$\rho_1 = \frac{ds}{d\tau} = \frac{ds^3}{R}.$$

The coördinates $x_0$, $y_0$, $z_0$, of the centre of the first curvature are

1911. $$x_0 = x + \rho_1^2 \frac{d\frac{dx}{ds}}{ds},$$

$$y_0 = y + \rho_1^2 \frac{d\frac{dy}{ds}}{ds},$$

$$z_0 = z + \rho_1^2 \frac{d\frac{dz}{ds}}{ds}.$$

Let the angle between two consecutive osculating planes be denoted by $d\theta$; then

1912. $$d\theta = \sqrt{(d\cos\lambda)^2 + (d\cos\mu)^2 + (d\cos\nu)^2}$$

$$= \frac{Ad^3x + Bd^3y + Cd^3z}{R^2}ds,$$

and the radius of the so-called *second curvature*, or *torsion* of the curve, is

1913. $$\rho_2 = \frac{ds}{d\theta}.$$

The curve is a plane curve (no torsion) when for all points $d\theta = 0$, that is, when

1914. $$Ad^3x + Bd^3y + Cd^3z = \begin{vmatrix} dx & dy & dz \\ d^2x & d^2y & d^2z \\ d^3x & d^3y & d^3z \end{vmatrix} = 0.$$

*The Helix.*

The helix is the curve formed by the thread of a screw, and may be defined as the locus of a point wnich moves with uniform velocity in the circumference of a circle while the circle itself moves with uniform velocity so as to generate a right circular cylinder.

Let the radius of the circle, also of the cylinder, $= r$; let the height through which the generating point rises during one passage around the circle (or cylinder) $= h$ ($=$ the interval between the threads of the screw); let $\alpha$ be the angle which the tangent to the curve makes with the plane of the generating circle, that is with the $xy$ plane; let the axis of the cylinder be the axis of $z$; and let $\varphi$ be the angle between the axis of $x$ and the projection of the radius vector on the $xy$ plane. Then

1915. $$\tan\alpha = \frac{h}{2\pi r}, \quad x = r\cos\varphi, \; y = r\sin\varphi, \; z = r\varphi\tan\alpha,$$

and the equations of the helix may be written

1916. $$x = r\cos\left(\frac{z}{r\tan\alpha}\right) \text{ and } y = r\sin\left(\frac{z}{r\tan\alpha}\right).$$

The equations of the tangent to the helix are

1917. $$-\frac{X - x}{\sin\varphi} = \frac{Y - y}{\cos\varphi} = \frac{Z - z}{\tan\alpha}.$$

If the surface of the cylinder be spread out on a plane the helix will become a straight line in that plane.

The radii of curvature are

1918. $$\rho_1 = r\,(1 + \tan^2\alpha) = \frac{r}{\cos^2\alpha},$$

$$\frac{1}{\rho_1} = \text{the } \textit{First Curvature.}$$

1919. $$\rho_2 = r\,\frac{1+\tan^2\alpha}{\tan\alpha} = \frac{r}{\sin\alpha\cos\alpha},$$

$$\frac{1}{\rho_2} = \text{the } \textit{Second Curvature.}$$

Length of an arc of the curve

1920. $$s = \frac{r}{\cos\alpha}\,\varphi = r\,\varphi\sqrt{1+\tan^2\alpha}.$$

# TABLES

## TABLE I.

| No. | Square | Cube | Square Root | Cube Root | Cube Root of the Square | Reciprocal |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1.0000000 | 1.0000000 | 1.0000000 | 1.000000000 |
| 2 | 4 | 8 | 1.4142136 | 1.2599210 | 1.5874011 | 0.500000000 |
| 3 | 9 | 27 | 1.7320508 | 1.4422496 | 2.0800837 | 0.333333333 |
| 4 | 16 | 64 | 2.0000000 | 1.5874011 | 2.5198421 | 0.250000000 |
| 5 | 25 | 125 | 2.2360680 | 1.7099759 | 2.9240177 | 0.200000000 |
| 6 | 36 | 216 | 2.4494897 | 1.8171206 | 3.3019272 | 0.166666667 |
| 7 | 49 | 343 | 2.6457513 | 1.9129312 | 3.6593057 | 0.142857143 |
| 8 | 64 | 512 | 2.8284271 | 2.0000000 | 4.0000000 | 0.125000000 |
| 9 | 81 | 729 | 3.0000000 | 2.0800837 | 4.3267487 | 0.111111111 |
| 10 | 100 | 1000 | 3.1622777 | 2.1544347 | 4.6415888 | 0.100000000 |
| 11 | 121 | 1331 | 3.3166248 | 2.2239801 | 4.9460874 | 0.090909091 |
| 12 | 144 | 1728 | 3.4641016 | 2.2894286 | 5.2414828 | 0.083333333 |
| 13 | 169 | 2197 | 3.6055513 | 2.3513347 | 5.5287748 | 0.076923077 |
| 14 | 196 | 2744 | 3.7416574 | 2.4101422 | 5.8087857 | 0.071428571 |
| 15 | 225 | 3375 | 3.8729833 | 2.4662121 | 6.0822020 | 0.066666667 |
| 16 | 256 | 4096 | 4.0000000 | 2.5198421 | 6.3496042 | 0.062500000 |
| 17 | 289 | 4913 | 4.1231056 | 2.5712816 | 6.6114890 | 0.058823529 |
| 18 | 324 | 5832 | 4.2426407 | 2.6207414 | 6.8682855 | 0.055555556 |
| 19 | 361 | 6859 | 4.3588989 | 2.6684016 | 7.1203674 | 0.052631579 |
| 20 | 400 | 8000 | 4.4721360 | 2.7144177 | 7.3680630 | 0.050000000 |
| 21 | 441 | 9261 | 4.5825757 | 2.7589243 | 7.6116626 | 0.047619048 |
| 22 | 484 | 10648 | 4.6904158 | 2.8020393 | 7.8514244 | 0.045454545 |
| 23 | 529 | 12167 | 4.7958315 | 2.8438670 | 8.0875794 | 0.043478261 |
| 24 | 576 | 13824 | 4.8989795 | 2.8844991 | 8.3203353 | 0.041666667 |
| 25 | 625 | 15625 | 5.0000000 | 2.9240177 | 8.5498797 | 0.040000000 |
| 26 | 676 | 17576 | 5.0990195 | 2.9624960 | 8.7763830 | 0.038461538 |
| 27 | 729 | 19683 | 5.1961524 | 3.0000000 | 9.0000000 | 0.037037037 |
| 28 | 784 | 21952 | 5.2915026 | 3.0365889 | 9.2208726 | 0.035714286 |
| 29 | 841 | 24389 | 5.3851648 | 3.0723168 | 9.4391307 | 0.034482759 |
| 30 | 900 | 27000 | 5.4772256 | 3.1072325 | 9.6548938 | 0.033333333 |
| 31 | 961 | 29791 | 5.5677644 | 3.1413806 | 9.8682724 | 0.032258065 |
| 32 | 1024 | 32768 | 5.6568542 | 3.1748021 | 10.0793684 | 0.031250000 |
| 33 | 1089 | 35937 | 5.7445626 | 3.2075343 | 10.2882765 | 0.030303030 |
| 34 | 1156 | 39304 | 5.8309519 | 3.2396118 | 10.4950847 | 0.029411765 |
| 35 | 1225 | 42875 | 5.9160798 | 3.2710663 | 10.6998748 | 0.028571429 |
| 36 | 1296 | 46656 | 6.0000000 | 3.3019272 | 10.9027235 | 0.027777778 |
| 37 | 1369 | 50653 | 6.0827625 | 3.3322218 | 11.1037025 | 0.027027027 |
| 38 | 1444 | 54872 | 6.1644140 | 3.3619754 | 11.3028786 | 0.026315789 |
| 39 | 1521 | 59319 | 6.2449980 | 3.3912114 | 11.5003151 | 0.025641026 |
| 40 | 1600 | 64000 | 6.3245553 | 3.4199519 | 11.6960709 | 0.025000000 |
| 41 | 1681 | 68921 | 6.4031242 | 3.4482172 | 11.8902022 | 0.024390244 |
| 42 | 1764 | 74088 | 6.4807407 | 3.4760266 | 12.0827612 | 0.023809524 |
| 43 | 1849 | 79507 | 6.5574385 | 3.5033981 | 12.2737980 | 0.023255814 |
| 44 | 1936 | 85184 | 6.6332496 | 3.5303483 | 12.4633594 | 0.022727273 |
| 45 | 2025 | 91125 | 6.7082039 | 3.5568933 | 12.6514900 | 0.022222222 |
| 46 | 2116 | 97336 | 6.7823300 | 3.5830479 | 12.8382321 | 0.021739130 |
| 47 | 2209 | 103823 | 6.8556546 | 3.6088261 | 13.0236256 | 0.021276600 |
| 48 | 2304 | 110592 | 6.9282032 | 3.6342411 | 13.2077090 | 0.020833333 |
| 49 | 2401 | 117649 | 7.0000000 | 3.6593057 | 13.3905183 | 0.020408163 |
| 50 | 2500 | 125000 | 7.0710678 | 3.6840314 | 13.5720880 | 0.020000000 |

TABLE I.

| No. | Square | Cube | Square Root | Cube Root | Cube Root of the Square | Reciprocal |
|---|---|---|---|---|---|---|
| 51 | 2601 | 132651 | 7.1414284 | 3.7084298 | 13.7524514 | 0.019607843 |
| 52 | 2704 | 140608 | 7.2111026 | 3.7325111 | 13.9316395 | 0.019230769 |
| 53 | 2809 | 148877 | 7.2801099 | 3.7562858 | 14.1096827 | 0.018867925 |
| 54 | 2916 | 157464 | 7.3484692 | 3.7797631 | 14.2866095 | 0.018518519 |
| 55 | 3025 | 166375 | 7.4161985 | 3.8029525 | 14.4624474 | 0.018181818 |
| 56 | 3136 | 175616 | 7.4833148 | 3.8258624 | 14.6372228 | 0.017857143 |
| 57 | 3249 | 185193 | 7.5498344 | 3.8485011 | 14.8109610 | 0.017543860 |
| 58 | 3364 | 195112 | 7.6157731 | 3.8708766 | 14.9836859 | 0.017241379 |
| 59 | 3481 | 205379 | 7.6811457 | 3.8929965 | 15.1554212 | 0.016949153 |
| 60 | 3600 | 216000 | 7.7459667 | 3.9148676 | 15.3261887 | 0.016666667 |
| 61 | 3721 | 226981 | 7.8102497 | 3.9364972 | 15.4960101 | 0.016393443 |
| 62 | 3844 | 238328 | 7.8740079 | 3.9578915 | 15.6649060 | 0.016129032 |
| 63 | 3969 | 250047 | 7.9372539 | 3.9790571 | 15.8328962 | 0.015873016 |
| 64 | 4096 | 262144 | 8.0000000 | 4.0000000 | 16.0000000 | 0.015625000 |
| 65 | 4225 | 274625 | 8.0622577 | 4.0207256 | 16.1662356 | 0.015384615 |
| 66 | 4356 | 287496 | 8.1240384 | 4.0412401 | 16.3316209 | 0.015151515 |
| 67 | 4489 | 300763 | 8.1853528 | 4.0615480 | 16.4961730 | 0.014925373 |
| 68 | 4624 | 314432 | 8.2462113 | 4.0816551 | 16.6599083 | 0.014705882 |
| 69 | 4761 | 328509 | 8.3066239 | 4.1015661 | 16.8228430 | 0.014492754 |
| 70 | 4900 | 343000 | 8.3666003 | 4.1212853 | 16.9849925 | 0.014285714 |
| 71 | 5041 | 357911 | 8.4261498 | 4.1408178 | 17.1463716 | 0.014084507 |
| 72 | 5184 | 373248 | 8.4852814 | 4.1601676 | 17.3069948 | 0.013888889 |
| 73 | 5329 | 389017 | 8.5440037 | 4.1793392 | 17.4668762 | 0.013698630 |
| 74 | 5476 | 405224 | 8.6023253 | 4.1983364 | 17.6260290 | 0.013513514 |
| 75 | 5625 | 421875 | 8.6602540 | 4.2171633 | 17.7844665 | 0.013333333 |
| 76 | 5776 | 438976 | 8.7177979 | 4.2358236 | 17.9422014 | 0.013157895 |
| 77 | 5929 | 456533 | 8.7749644 | 4.2543210 | 18.0992460 | 0.012987013 |
| 78 | 6084 | 474552 | 8.8317609 | 4.2726586 | 18.2556122 | 0.012820513 |
| 79 | 6241 | 493039 | 8.8881944 | 4.2908404 | 18.4113116 | 0.012658228 |
| 80 | 6400 | 512000 | 8.9442719 | 4.3088695 | 18.5663553 | 0.012500000 |
| 81 | 6561 | 531441 | 9.0000000 | 4.3267487 | 18.7207544 | 0.012345679 |
| 82 | 6724 | 551368 | 9.0553851 | 4.3444815 | 18.8745194 | 0.012195122 |
| 83 | 6889 | 571787 | 9.1104336 | 4.3620707 | 19.0276606 | 0.012048193 |
| 84 | 7056 | 592704 | 9.1651514 | 4.3795191 | 19.1801879 | 0.011904762 |
| 85 | 7225 | 614125 | 9.2195445 | 4.3968296 | 19.3321111 | 0.011764706 |
| 86 | 7396 | 636056 | 9.2736185 | 4.4140049 | 19.4834398 | 0.011627907 |
| 87 | 7569 | 658503 | 9.3273791 | 4.4310476 | 19.6341830 | 0.011494253 |
| 88 | 7744 | 681472 | 9.3808315 | 4.4479602 | 19.7843498 | 0.011363636 |
| 89 | 7921 | 704969 | 9.4339811 | 4.4647451 | 19.9339487 | 0.011235955 |
| 90 | 8100 | 729000 | 9.4868330 | 4.4814047 | 20.0829885 | 0.011111111 |
| 91 | 8281 | 753571 | 9.5393920 | 4.4979414 | 20.2314773 | 0.010989011 |
| 92 | 8464 | 778688 | 9.5916630 | 4.5143574 | 20.3794231 | 0.010869565 |
| 93 | 8649 | 804357 | 9.6436508 | 4.5306549 | 20.5268337 | 0.010752688 |
| 94 | 8836 | 830584 | 9.6953597 | 4.5468359 | 20.6737171 | 0.010638298 |
| 95 | 9025 | 857375 | 9.7467943 | 4.5629026 | 20.8200805 | 0.010526316 |
| 96 | 9216 | 884736 | 9.7979590 | 4.5788570 | 20.9659311 | 0.010416667 |
| 97 | 9409 | 912673 | 9.8488578 | 4.5947009 | 21.1112763 | 0.010309278 |
| 98 | 9604 | 941192 | 9.8994949 | 4.6104363 | 21.2561227 | 0.010204082 |
| 99 | 9801 | 970299 | 9.9498744 | 4.6260650 | 21.4004775 | 0.010101010 |
| 100 | 10000 | 1000000 | 10.0000000 | 4.6415888 | 21.5443469 | 0.010000000 |

## TABLE I.

| No. | Square | Cube | Square Root | Cube Root | Cube Root of the Square | Reciprocal 0.00 |
|---|---|---|---|---|---|---|
| 101 | 10201 | 1030301 | 10.0498756 | 4.6570095 | 21.68774 | 9900990 |
| 102 | 10404 | 1061208 | 10.0995049 | 4.6723287 | 21.83066 | 9803922 |
| 103 | 10609 | 1092727 | 10.1488916 | 4.6875482 | 21.97311 | 9708738 |
| 104 | 10816 | 1124864 | 10.1980390 | 4.7026694 | 22.11510 | 9615385 |
| 105 | 11025 | 1157625 | 10.2469508 | 4.7176940 | 22.25664 | 9523810 |
| 106 | 11236 | 1191016 | 10.2956301 | 4.7326235 | 22.39773 | 9433962 |
| 107 | 11449 | 1225043 | 10.3440804 | 4.7474594 | 22.53837 | 9345794 |
| 108 | 11664 | 1259712 | 10.3923048 | 4.7622032 | 22.67858 | 9259259 |
| 109 | 11881 | 1295029 | 10.4403065 | 4.7768562 | 22.81835 | 9174312 |
| 110 | 12100 | 1331000 | 10.4880885 | 4.7914199 | 22.95770 | 9090909 |
| 111 | 12321 | 1367631 | 10.5356538 | 4.8058955 | 23.09663 | 9009009 |
| 112 | 12544 | 1404928 | 10.5830052 | 4.8202845 | 23.23514 | 8928571 |
| 113 | 12769 | 1442897 | 10.6301458 | 4.8345881 | 23.37324 | 8849558 |
| 114 | 12996 | 1481544 | 10.6770783 | 4.8488076 | 23.51094 | 8771930 |
| 115 | 13225 | 1520875 | 10.7238053 | 4.8629442 | 23.64823 | 8695652 |
| 116 | 13456 | 1560896 | 10.7703296 | 4.8769990 | 23.78512 | 8620690 |
| 117 | 13689 | 1601613 | 10.8166538 | 4.8909732 | 23.92162 | 8547009 |
| 118 | 13924 | 1643032 | 10.8627805 | 4.9048681 | 24.05773 | 8474576 |
| 119 | 14161 | 1685159 | 10.9087121 | 4.9186847 | 24.19346 | 8403361 |
| 120 | 14400 | 1728000 | 10.9544512 | 4.9324242 | 24.32881 | 8333333 |
| 121 | 14641 | 1771561 | 11.0000000 | 4.9460874 | 24.46378 | 8264463 |
| 122 | 14884 | 1815848 | 11.0453610 | 4.9596757 | 24.59838 | 8196721 |
| 123 | 15129 | 1860867 | 11.0905365 | 4.9731898 | 24.73262 | 8130081 |
| 124 | 15376 | 1906624 | 11.1355287 | 4.9866310 | 24.86649 | 8064516 |
| 125 | 15625 | 1953125 | 11.1803399 | 5.0000000 | 25.00000 | 8000000 |
| 126 | 15876 | 2000376 | 11.2249722 | 5.0132979 | 25.13316 | 7936508 |
| 127 | 16129 | 2048383 | 11.2694277 | 5.0265257 | 25.26596 | 7874016 |
| 128 | 16384 | 2097152 | 11.3137085 | 5.0396842 | 25.39842 | 7812500 |
| 129 | 16641 | 2146689 | 11.3578167 | 5.0527743 | 25.53053 | 7751938 |
| 130 | 16900 | 2197000 | 11.4017543 | 5.0657970 | 25.66230 | 7692308 |
| 131 | 17161 | 2248091 | 11.4455231 | 5.0787531 | 25.79373 | 7633588 |
| 132 | 17424 | 2299968 | 11.4891253 | 5.0916434 | 25.92483 | 7575758 |
| 133 | 17689 | 2352637 | 11.5325626 | 5.1044687 | 26.05560 | 7518797 |
| 134 | 17956 | 2406104 | 11.5758369 | 5.1172299 | 26.18604 | 7462687 |
| 135 | 18225 | 2460375 | 11.6189500 | 5.1299278 | 26.31616 | 7407407 |
| 136 | 18496 | 2515456 | 11.6619038 | 5.1425632 | 26.44596 | 7352941 |
| 137 | 18769 | 2571353 | 11.7046999 | 5.1551367 | 26.57543 | 7299270 |
| 138 | 19044 | 2628072 | 11.7473401 | 5.1676493 | 26.70460 | 7246377 |
| 139 | 19321 | 2685619 | 11.7898261 | 5.1801015 | 26.83345 | 7194245 |
| 140 | 19600 | 2744000 | 11.8321596 | 5.1924941 | 26.96199 | 7142857 |
| 141 | 19881 | 2803221 | 11.8743422 | 5.2048279 | 27.09023 | 7092199 |
| 142 | 20164 | 2863288 | 11.9163753 | 5.2171034 | 27.21817 | 7042254 |
| 143 | 20449 | 2924207 | 11.9582607 | 5.2293215 | 27.34580 | 6993007 |
| 144 | 20736 | 2985984 | 12.0000000 | 5.2414828 | 27.47314 | 6944444 |
| 145 | 21025 | 3048625 | 12.0415946 | 5.2535879 | 27.60019 | 6896552 |
| 146 | 21316 | 3112136 | 12.0830460 | 5.2656374 | 27.72694 | 6849315 |
| 147 | 21609 | 3176523 | 12.1243557 | 5.2776321 | 27.85340 | 6802721 |
| 148 | 21904 | 3241792 | 12.1655251 | 5.2895725 | 27.97958 | 6756757 |
| 149 | 22201 | 3307949 | 12.2065556 | 5.3014592 | 28.10547 | 6711409 |
| 150 | 22500 | 3375000 | 12.2474487 | 5.3132928 | 28.23108 | 6666667 |

# TABLE I.

| No. | Square | Cube | Square Root | Cube Root | Cube Root of the Square | Reciprocal 0.00 |
|---|---|---|---|---|---|---|
| 151 | 22801 | 3442951 | 12.2882057 | 5.3250740 | 28.35641 | 6622517 |
| 152 | 23104 | 3511808 | 12.3288280 | 5.3368033 | 28.48147 | 6578947 |
| 153 | 23409 | 3581577 | 12.3693169 | 5.3484812 | 28.60625 | 6535948 |
| 154 | 23716 | 3652264 | 12.4096736 | 5.3601084 | 28.73076 | 6493506 |
| 155 | 24025 | 3723875 | 12.4498996 | 5.3716854 | 28.85500 | 6451613 |
| 156 | 24336 | 3796416 | 12.4899960 | 5.3832126 | 28.97898 | 6410256 |
| 157 | 24649 | 3869893 | 12.5299641 | 5.3946907 | 29.10269 | 6369427 |
| 158 | 24964 | 3944312 | 12.5698051 | 5.4061202 | 29.22614 | 6329114 |
| 159 | 25281 | 4019679 | 12.6095202 | 5.4175015 | 29.34932 | 6289308 |
| 160 | 25600 | 4096000 | 12.6491106 | 5.4288352 | 29.47225 | 6250000 |
| 161 | 25921 | 4173281 | 12.6885775 | 5.4401218 | 29.59493 | 6211180 |
| 162 | 26244 | 4251528 | 12.7279221 | 5.4513618 | 29.71735 | 6172840 |
| 163 | 26569 | 4330747 | 12.7671453 | 5.4625556 | 29.83951 | 6134969 |
| 164 | 26896 | 4410944 | 12.8062485 | 5.4737037 | 29.96143 | 6097561 |
| 165 | 27225 | 4492125 | 12.8452326 | 5.4848066 | 30.08310 | 6060606 |
| 166 | 27556 | 4574296 | 12.8840987 | 5.4958647 | 30.20453 | 6024096 |
| 167 | 27889 | 4657463 | 12.9228480 | 5.5068784 | 30.32571 | 5988024 |
| 168 | 28224 | 4741632 | 12.9614814 | 5.5178484 | 30.44665 | 5952381 |
| 169 | 28561 | 4826809 | 13.0000000 | 5.5287748 | 30.56735 | 5917160 |
| 170 | 28900 | 4913000 | 13.0384048 | 5.5396583 | 30.68781 | 5882353 |
| 171 | 29241 | 5000211 | 13.0766968 | 5.5504991 | 30.80804 | 5847953 |
| 172 | 29584 | 5088448 | 13.1148770 | 5.5612978 | 30.92803 | 5813953 |
| 173 | 29929 | 5177717 | 13.1529464 | 5.5720546 | 31.04779 | 5780347 |
| 174 | 30276 | 5268024 | 13.1909060 | 5.5827702 | 31.16732 | 5747126 |
| 175 | 30625 | 5359375 | 13.2287566 | 5.5934447 | 31.28662 | 5714286 |
| 176 | 30976 | 5451776 | 13.2664992 | 5.6040787 | 31.40570 | 5681818 |
| 177 | 31329 | 5545233 | 13.3041347 | 5.6146724 | 31.52455 | 5649718 |
| 178 | 31684 | 5639752 | 13.3416641 | 5.6252263 | 31.64317 | 5617978 |
| 179 | 32041 | 5735339 | 13.3790882 | 5.6357408 | 31.76157 | 5586592 |
| 180 | 32400 | 5832000 | 13.4164079 | 5.6462162 | 31.87976 | 5555556 |
| 181 | 32761 | 5929741 | 13.4536240 | 5.6566528 | 31.99772 | 5524862 |
| 182 | 33124 | 6028568 | 13.4907376 | 5.6670511 | 32.11547 | 5494505 |
| 183 | 33489 | 6128487 | 13.5277493 | 5.6774114 | 32.23300 | 5464481 |
| 184 | 33856 | 6229504 | 13.5646600 | 5.6877340 | 32.35032 | 5434783 |
| 185 | 34225 | 6331625 | 13.6014705 | 5.6980192 | 32.46742 | 5405405 |
| 186 | 34596 | 6434856 | 13.6381817 | 5.7082675 | 32.58432 | 5376344 |
| 187 | 34969 | 6539203 | 13.6747943 | 5.7184791 | 32.70100 | 5347594 |
| 188 | 35344 | 6644672 | 13.7113092 | 5.7286543 | 32.81748 | 5319149 |
| 189 | 35721 | 6751269 | 13.7477271 | 5.7387936 | 32.93375 | 5291005 |
| 190 | 36100 | 6859000 | 13.7840488 | 5.7488971 | 33.04982 | 5263158 |
| 191 | 36481 | 6967871 | 13.8202750 | 5.7589652 | 33.16568 | 5235602 |
| 192 | 36864 | 7077888 | 13.8564065 | 5.7689982 | 33.28134 | 5208333 |
| 193 | 37249 | 7189057 | 13.8924440 | 5.7789966 | 33.39680 | 5181347 |
| 194 | 37636 | 7301384 | 13.9283883 | 5.7889604 | 33.51206 | 5154639 |
| 195 | 38025 | 7414875 | 13.9642400 | 5.7988900 | 33.62713 | 5128205 |
| 196 | 38416 | 7529536 | 14.0000000 | 5.8087857 | 33.74199 | 5102041 |
| 197 | 38809 | 7645373 | 14.0356688 | 5.8186479 | 33.85666 | 5076142 |
| 198 | 39204 | 7762392 | 14.0712473 | 5.8284767 | 33.97114 | 5050505 |
| 199 | 39601 | 7880599 | 14.1067360 | 5.8382725 | 34.08543 | 5025126 |
| 200 | 40000 | 8000000 | 14.1421356 | 5.8480355 | 34.19952 | 5000000 |

## TABLE I.

| No. | Square | Cube | Square Root | Cube Root | Cube Root of the Square | Reciprocal 0.00 |
|---|---|---|---|---|---|---|
| **201** | 40401 | 8120601 | 14.1774469 | 5.8577660 | 34.31342 | 4975124 |
| **202** | 40804 | 8242408 | 14.2126704 | 5.8674643 | 34.42714 | 4950495 |
| **203** | 41209 | 8365427 | 14.2478068 | 5.8771307 | 34.54066 | 4926108 |
| **204** | 41616 | 8489664 | 14.2828569 | 5.8867653 | 34.65400 | 4901961 |
| **205** | 42025 | 8615125 | 14.3178211 | 5.8963685 | 34.76716 | 4878049 |
| **206** | 42436 | 8741816 | 14.3527001 | 5.9059406 | 34.88013 | 4854369 |
| **207** | 42849 | 8869743 | 14.3874946 | 5.9154817 | 34.99292 | 4830918 |
| **208** | 43264 | 8998912 | 14.4222051 | 5.9249921 | 35.10553 | 4807692 |
| **209** | 43681 | 9129329 | 14.4568323 | 5.9344721 | 35.21796 | 4784689 |
| **210** | 44100 | 9261000 | 14.4913767 | 5.9439220 | 35.33021 | 4761905 |
| **211** | 44521 | 9393931 | 14.5258390 | 5.9533418 | 35.44228 | 4739336 |
| **212** | 44944 | 9528128 | 14.5602198 | 5.9627320 | 35.55417 | 4716981 |
| **213** | 45369 | 9663597 | 14.5945195 | 5.9720926 | 35.66589 | 4694836 |
| **214** | 45796 | 9800344 | 14.6287388 | 5.9814240 | 35.77743 | 4672897 |
| **215** | 46225 | 9938375 | 14.6628783 | 5.9907264 | 35.88880 | 4651163 |
| **216** | 46656 | 10077696 | 14.6969385 | 6.0000000 | 36.00000 | 4629630 |
| **217** | 47089 | 10218313 | 14.7309199 | 6.0092450 | 36.11103 | 4608295 |
| **218** | 47524 | 10360232 | 14.7648231 | 6.0184617 | 36.22188 | 4587156 |
| **219** | 47961 | 10503459 | 14.7986486 | 6.0276502 | 36.33257 | 4566210 |
| **220** | 48400 | 10648000 | 14.8323970 | 6.0368107 | 36.44308 | 4545455 |
| **221** | 48841 | 10793861 | 14.8660687 | 6.0459435 | 36.55343 | 4524887 |
| **222** | 49284 | 10941048 | 14.8996644 | 6.0550489 | 36.66362 | 4504505 |
| **223** | 49729 | 11089567 | 14.9331845 | 6.0641270 | 36.77364 | 4484305 |
| **224** | 50176 | 11239424 | 14.9666295 | 6.0731779 | 36.88349 | 4464286 |
| **225** | 50625 | 11390625 | 15.0000000 | 6.0822020 | 36.99318 | 4444444 |
| **226** | 51076 | 11543176 | 15.0332964 | 6.0911994 | 37.10271 | 4424779 |
| **227** | 51529 | 11697083 | 15.0665192 | 6.1001702 | 37.21208 | 4405286 |
| **228** | 51984 | 11852352 | 15.0996689 | 6.1091147 | 37.32128 | 4385965 |
| **229** | 52441 | 12008989 | 15.1327460 | 6.1180332 | 37.43033 | 4366812 |
| **230** | 52900 | 12167000 | 15.1657509 | 6.1269257 | 37.53922 | 4347826 |
| **231** | 53361 | 12326391 | 15.1986842 | 6.1357924 | 37.64795 | 4329004 |
| **232** | 53824 | 12487168 | 15.2315462 | 6.1446337 | 37.75652 | 4310345 |
| **233** | 54289 | 12649337 | 15.2643375 | 6.1534495 | 37.86494 | 4291845 |
| **234** | 54756 | 12812904 | 15.2970585 | 6.1622401 | 37.97320 | 4273504 |
| **235** | 55225 | 12977875 | 15.3297097 | 6.1710058 | 38.08131 | 4255319 |
| **236** | 55696 | 13144256 | 15.3622915 | 6.1797466 | 38.18927 | 4237288 |
| **237** | 56169 | 13312053 | 15.3948043 | 6.1884628 | 38.29707 | 4219409 |
| **238** | 56644 | 13481272 | 15.4272486 | 6.1971544 | 38.40472 | 4201681 |
| **239** | 57121 | 13651919 | 15.4596248 | 6.2058218 | 38.51223 | 4184100 |
| **240** | 57600 | 13824000 | 15.4919334 | 6.2144650 | 38.61958 | 4166667 |
| **241** | 58081 | 13997521 | 15.5241747 | 6.2230843 | 38.72678 | 4149378 |
| **242** | 58564 | 14172488 | 15.5563492 | 6.2316797 | 38.83383 | 4132231 |
| **243** | 59049 | 14348907 | 15.5884573 | 6.2402515 | 38.94074 | 4115226 |
| **244** | 59536 | 14526784 | 15.6204994 | 6.2487998 | 39.04750 | 4098361 |
| **245** | 60025 | 14706125 | 15.6524758 | 6.2573248 | 39.15411 | 4081633 |
| **246** | 60516 | 14886936 | 15.6843871 | 6.2658266 | 39.26058 | 4065041 |
| **247** | 61009 | 15069223 | 15.7162336 | 6.2743054 | 39.36691 | 4048583 |
| **248** | 61504 | 15252992 | 15.7480157 | 6.2827613 | 39.47309 | 4032258 |
| **249** | 62001 | 15438249 | 15.7797338 | 6.2911946 | 39.57913 | 4016064 |
| **250** | 62500 | 15625000 | 15.8113883 | 6.2996053 | 39.68503 | 4000000 |

TABLE I.

| No. | Square | Cube | Square Root | Cube Root | Cube Root of the Square | Reciprocal 0.00 |
|---|---|---|---|---|---|---|
| 251 | 63001 | 15813251 | 15.8429795 | 6.3079935 | 39.79078 | 3984064 |
| 252 | 63504 | 16003008 | 15.8745079 | 6.3163596 | 39.89640 | 3968254 |
| 253 | 64009 | 16194277 | 15.9059737 | 6.3247035 | 40.00187 | 3952569 |
| 254 | 64516 | 16387064 | 15.9373775 | 6.3330256 | 40.10721 | 3937008 |
| 255 | 65025 | 16581375 | 15.9687194 | 6.3413257 | 40.21241 | 3921569 |
| 256 | 65536 | 16777216 | 16.0000000 | 6.3496042 | 40.31747 | 3906250 |
| 257 | 66049 | 16974593 | 16.0312195 | 6.3578611 | 40.42240 | 3891051 |
| 258 | 66564 | 17173512 | 16.0623784 | 6.3660968 | 40.52719 | 3875969 |
| 259 | 67081 | 17373979 | 16.0934769 | 6.3743111 | 40.63184 | 3861004 |
| 260 | 67600 | 17576000 | 16.1245155 | 6.3825043 | 40.73636 | 3846154 |
| 261 | 68121 | 17779581 | 16.1554944 | 6.3906765 | 40.84075 | 3831418 |
| 262 | 68644 | 17984728 | 16.1864141 | 6.3988279 | 40.94500 | 3816794 |
| 263 | 69169 | 18191447 | 16.2172747 | 6.4069585 | 41.04912 | 3802281 |
| 264 | 69696 | 18399744 | 16.2480768 | 6.4150687 | 41.15311 | 3787879 |
| 265 | 70225 | 18609625 | 16.2788206 | 6.4231583 | 41.25696 | 3773585 |
| 266 | 70756 | 18821096 | 16.3095064 | 6.4312276 | 41.36069 | 3759398 |
| 267 | 71289 | 19034163 | 16.3401346 | 6.4392767 | 41.46428 | 3745318 |
| 268 | 71824 | 19248832 | 16.3707055 | 6.4473057 | 41.56775 | 3731343 |
| 269 | 72361 | 19465109 | 16.4012195 | 6.4553148 | 41.67109 | 3717472 |
| 270 | 72900 | 19683000 | 16.4316767 | 6.4633041 | 41.77430 | 3703704 |
| 271 | 73441 | 19902511 | 16.4620776 | 6.4712736 | 41.87738 | 3690037 |
| 272 | 73984 | 20123648 | 16.4924225 | 6.4792236 | 41.98034 | 3676471 |
| 273 | 74529 | 20346417 | 16.5227116 | 6.4871541 | 42.08317 | 3663004 |
| 274 | 75076 | 20570824 | 16.5529454 | 6.4950653 | 42.18587 | 3649635 |
| 275 | 75625 | 20796875 | 16.5831240 | 6.5029572 | 42.28845 | 3636364 |
| 276 | 76176 | 21024576 | 16.6132477 | 6.5108300 | 42.39091 | 3623188 |
| 277 | 76729 | 21253933 | 16.6433170 | 6.5186839 | 42.49324 | 3610108 |
| 278 | 77284 | 21484952 | 16.6733320 | 6.5265189 | 42.59545 | 3597122 |
| 279 | 77841 | 21717639 | 16.7032931 | 6.5343351 | 42.69753 | 3584229 |
| 280 | 78400 | 21952000 | 16.7332005 | 6.5421326 | 42.79950 | 3571429 |
| 281 | 78961 | 22188041 | 16.7630546 | 6.5499116 | 42.90134 | 3558719 |
| 282 | 79524 | 22425768 | 16.7928556 | 6.5576722 | 43.00306 | 3546099 |
| 283 | 80089 | 22665187 | 16.8226038 | 6.5654144 | 43.10467 | 3533569 |
| 284 | 80656 | 22906304 | 16.8522995 | 6.5731385 | 43.20615 | 3521127 |
| 285 | 81225 | 23149125 | 16.8819430 | 6.5808443 | 43.30751 | 3508772 |
| 286 | 81796 | 23393656 | 16.9115345 | 6.5885323 | 43.40876 | 3496503 |
| 287 | 82369 | 23639903 | 16.9410743 | 6.5962023 | 43.50988 | 3484321 |
| 288 | 82944 | 23887872 | 16.9705627 | 6.6038545 | 43.61089 | 3472222 |
| 289 | 83521 | 24137569 | 17.0000000 | 6.6114890 | 43.71179 | 3460208 |
| 290 | 84100 | 24389000 | 17.0293864 | 6.6191060 | 43.81256 | 3448276 |
| 291 | 84681 | 24642171 | 17.0587221 | 6.6267054 | 43.91322 | 3436426 |
| 292 | 85264 | 24897088 | 17.0880075 | 6.6342874 | 44.01377 | 3424658 |
| 293 | 85849 | 25153757 | 17.1172428 | 6.6418522 | 44.11420 | 3412969 |
| 294 | 86436 | 25412184 | 17.1464282 | 6.6493998 | 44.21452 | 3401361 |
| 295 | 87025 | 25672375 | 17.1755640 | 6.6569302 | 44.31472 | 3389831 |
| 296 | 87616 | 25934336 | 17.2046505 | 6.6644437 | 44.41481 | 3378378 |
| 297 | 88209 | 26198073 | 17.2336879 | 6.6719403 | 44.51479 | 3367003 |
| 298 | 88804 | 26463592 | 17.2626765 | 6.6794200 | 44.61465 | 3355705 |
| 299 | 89401 | 26730899 | 17.2916165 | 6.6868831 | 44.71441 | 3344482 |
| 300 | 90000 | 27000000 | 17.3205081 | 6.6943295 | 44.81405 | 3333333 |

TABLE I.

| No. | Square | Cube | Square Root | Cube Root | Cube Root of the Square | Reciprocal 0.00 |
|---|---|---|---|---|---|---|
| 301 | 90601 | 27270901 | 17.3493516 | 6.7017593 | 44.91358 | 3322259 |
| 302 | 91204 | 27543608 | 17.3781472 | 6.7091729 | 45.01300 | 3311258 |
| 303 | 91809 | 27818127 | 17.4068952 | 6.7165700 | 45.11231 | 3300330 |
| 304 | 92416 | 28094464 | 17.4355958 | 6.7239508 | 45.21151 | 3289474 |
| 305 | 93025 | 28372625 | 17.4642492 | 6.7313155 | 45.31061 | 3278689 |
| 306 | 93636 | 28652616 | 17.4928557 | 6.7386641 | 45.40959 | 3267974 |
| 307 | 94249 | 28934443 | 17.5214155 | 6.7459967 | 45.50847 | 3257329 |
| 308 | 94864 | 29218112 | 17.5499288 | 6.7533134 | 45.60725 | 3246753 |
| 309 | 95481 | 29503629 | 17.5783958 | 6.7606143 | 45.70591 | 3236246 |
| 310 | 96100 | 29791000 | 17.6068169 | 6.7678995 | 45.80446 | 3225806 |
| 311 | 96721 | 30080231 | 17.6351921 | 6.7751690 | 45.90291 | 3215434 |
| 312 | 97344 | 30371328 | 17.6635217 | 6.7824229 | 46.00126 | 3205128 |
| 313 | 97969 | 30664297 | 17.6918060 | 6.7896613 | 46.09950 | 3194888 |
| 314 | 98596 | 30959144 | 17.7200451 | 6.7968844 | 46.19764 | 3184713 |
| 315 | 99225 | 31255875 | 17.7482393 | 6.8040921 | 46.29566 | 3174603 |
| 316 | 99856 | 31554496 | 17.7763888 | 6.8112847 | 46.39360 | 3164557 |
| 317 | 100489 | 31855013 | 17.8044938 | 6.8184620 | 46.49142 | 3154574 |
| 318 | 101124 | 32157432 | 17.8325545 | 6.8256242 | 46.58915 | 3144654 |
| 319 | 101761 | 32461759 | 17.8605711 | 6.8327714 | 46.68677 | 3134796 |
| 320 | 102400 | 32768000 | 17.8885438 | 6.8399037 | 46.78428 | 3125000 |
| 321 | 103041 | 33076161 | 17.9164729 | 6.8470213 | 46.88170 | 3115265 |
| 322 | 103684 | 33386248 | 17.9443584 | 6.8541240 | 46.97902 | 3105590 |
| 323 | 104329 | 33698267 | 17.9722008 | 6.8612120 | 47.07623 | 3095975 |
| 324 | 104976 | 34012224 | 18.0000000 | 6.8682855 | 47.17335 | 3086420 |
| 325 | 105625 | 34328125 | 18.0277564 | 6.8753443 | 47.27036 | 3076923 |
| 326 | 106276 | 34645976 | 18.0554701 | 6.8823888 | 47.36727 | 3067485 |
| 327 | 106929 | 34965783 | 18.0831413 | 6.8894188 | 47.46409 | 3058104 |
| 328 | 107584 | 35287552 | 18.1107703 | 6.8964345 | 47.56081 | 3048780 |
| 329 | 108241 | 35611289 | 18.1383571 | 6.9034359 | 47.65743 | 3039514 |
| 330 | 108900 | 35937000 | 18.1659021 | 6.9104232 | 47.75395 | 3030303 |
| 331 | 109561 | 36264691 | 18.1934054 | 6.9173964 | 47.85037 | 3021148 |
| 332 | 110224 | 36594368 | 18.2208672 | 6.9243556 | 47.94670 | 3012048 |
| 333 | 110889 | 36926037 | 18.2482876 | 6.9313008 | 48.04293 | 3003003 |
| 334 | 111556 | 37259704 | 18.2756669 | 6.9382321 | 48.13906 | 2994012 |
| 335 | 112225 | 37595375 | 18.3030052 | 6.9451496 | 48.23510 | 2985075 |
| 336 | 112896 | 37933056 | 18.3303028 | 6.9520533 | 48.33104 | 2976190 |
| 337 | 113569 | 38272753 | 18.3575598 | 6.9589434 | 48.42689 | 2967359 |
| 338 | 114244 | 38614472 | 18.3847763 | 6.9658198 | 48.52265 | 2958580 |
| 339 | 114921 | 38958219 | 18.4119526 | 6.9726826 | 48.61830 | 2949853 |
| 340 | 115600 | 39304000 | 18.4390889 | 6.9795321 | 48.71387 | 2941176 |
| 341 | 116281 | 39651821 | 18.4661853 | 6.9863681 | 48.80934 | 2932551 |
| 342 | 116964 | 40001688 | 18.4932420 | 6.9931906 | 48.90473 | 2923977 |
| 343 | 117649 | 40353607 | 18.5202592 | 7.0000000 | 49.00000 | 2915452 |
| 344 | 118336 | 40707584 | 18.5472370 | 7.0067962 | 49.09519 | 2906977 |
| 345 | 119025 | 41063625 | 18.5741756 | 7.0135791 | 49.19029 | 2898551 |
| 346 | 119716 | 41421736 | 18.6010752 | 7.0203490 | 49.28530 | 2890173 |
| 347 | 120409 | 41781923 | 18.6279360 | 7.0271058 | 49.38022 | 2881844 |
| 348 | 121104 | 42144192 | 18.6547581 | 7.0338497 | 49.47504 | 2873563 |
| 349 | 121801 | 42508549 | 18.6815417 | 7.0405806 | 49.56978 | 2865330 |
| 350 | 122500 | 42875000 | 18.7082869 | 7.0472987 | 49.66442 | 2857143 |

## TABLE I.

| No. | Square | Cube | Square Root | Cube Root | Cube Root of the Square | Reciprocal 0.00 |
|---|---|---|---|---|---|---|
| 351 | 123201 | 43243551 | 18.7349940 | 7.0540041 | 49.75898 | 2849003 |
| 352 | 123904 | 43614208 | 18.7616630 | 7.0606967 | 49.85344 | 2840909 |
| 353 | 124609 | 43986977 | 18.7882942 | 7.0673767 | 49.94781 | 2832861 |
| 354 | 125316 | 44361864 | 18.8148877 | 7.0740440 | 50.04209 | 2824859 |
| 355 | 126025 | 44738875 | 18.8414437 | 7.0806988 | 50.13629 | 2816901 |
| 356 | 126736 | 45118016 | 18.8679623 | 7.0873411 | 50.23040 | 2808989 |
| 357 | 127449 | 45499293 | 18.8944436 | 7.0939709 | 50.32442 | 2801120 |
| 358 | 128164 | 45882712 | 18.9208879 | 7.1005885 | 50.41836 | 2793296 |
| 359 | 128881 | 46268279 | 18.9472953 | 7.1071937 | 50.51220 | 2785515 |
| 360 | 129600 | 46656000 | 18.9736660 | 7.1137866 | 50.60596 | 2777778 |
| 361 | 130321 | 47045881 | 19.0000000 | 7.1203674 | 50.69963 | 2770083 |
| 362 | 131044 | 47437928 | 19.0262976 | 7.1269360 | 50.79322 | 2762431 |
| 363 | 131769 | 47832147 | 19.0525589 | 7.1334925 | 50.88672 | 2754821 |
| 364 | 132496 | 48228544 | 19.0787840 | 7.1400370 | 50.98013 | 2747253 |
| 365 | 133225 | 48627125 | 19.1049732 | 7.1465695 | 51.07346 | 2739726 |
| 366 | 133956 | 49027896 | 19.1311265 | 7.1530901 | 51.16670 | 2732240 |
| 367 | 134689 | 49430863 | 19.1572441 | 7.1595988 | 51.25986 | 2724796 |
| 368 | 135424 | 49836032 | 19.1833261 | 7.1660957 | 51.35293 | 2717391 |
| 369 | 136161 | 50243409 | 19.2093727 | 7.1725809 | 51.44592 | 2710027 |
| 370 | 136900 | 50653000 | 19.2353841 | 7.1790544 | 51.53882 | 2702703 |
| 371 | 137641 | 51064811 | 19.2613603 | 7.1855162 | 51.63164 | 2695418 |
| 372 | 138384 | 51478848 | 19.2873015 | 7.1919663 | 51.72438 | 2688172 |
| 373 | 139129 | 51895117 | 19.3132079 | 7.1984050 | 51.81703 | 2680965 |
| 374 | 139876 | 52313624 | 19.3390796 | 7.2048322 | 51.90961 | 2673797 |
| 375 | 140625 | 52734375 | 19.3649167 | 7.2112479 | 52.00210 | 2666667 |
| 376 | 141376 | 53157376 | 19.3907194 | 7.2176522 | 52.09450 | 2659574 |
| 377 | 142129 | 53582633 | 19.4164878 | 7.2240450 | 52.18683 | 2652520 |
| 378 | 142884 | 54010152 | 19.4422221 | 7.2304268 | 52.27907 | 2645503 |
| 379 | 143641 | 54439939 | 19.4679223 | 7.2367972 | 52.37123 | 2638522 |
| 380 | 144400 | 54872000 | 19.4935887 | 7.2431565 | 52.46332 | 2631579 |
| 381 | 145161 | 55306341 | 19.5192213 | 7.2495045 | 52.55532 | 2624672 |
| 382 | 145924 | 55742968 | 19.5448203 | 7.2558415 | 52.64724 | 2617801 |
| 383 | 146689 | 56181887 | 19.5703858 | 7.2621675 | 52.73908 | 2610966 |
| 384 | 147456 | 56623104 | 19.5959179 | 7.2684824 | 52.83084 | 2604167 |
| 385 | 148225 | 57066625 | 19.6214169 | 7.2747864 | 52.92252 | 2597403 |
| 386 | 148996 | 57512456 | 19.6468827 | 7.2810794 | 53.01412 | 2590674 |
| 387 | 149769 | 57960603 | 19.6723156 | 7.2873617 | 53.10564 | 2583979 |
| 388 | 150544 | 58411072 | 19.6977156 | 7.2936330 | 53.19708 | 2577320 |
| 389 | 151321 | 58863869 | 19.7230829 | 7.2998936 | 53.28845 | 2570694 |
| 390 | 152100 | 59319000 | 19.7484177 | 7.3061436 | 53.37973 | 2564103 |
| 391 | 152881 | 59776471 | 19.7737199 | 7.3123828 | 53.47094 | 2557545 |
| 392 | 153664 | 60236288 | 19.7989899 | 7.3186114 | 53.56207 | 2551020 |
| 393 | 154449 | 60698457 | 19.8242276 | 7.3248295 | 53.65313 | 2544529 |
| 394 | 155236 | 61162984 | 19.8494332 | 7.3310369 | 53.74410 | 2538071 |
| 395 | 156025 | 61629875 | 19.8746069 | 7.3372339 | 53.83500 | 2531646 |
| 396 | 156816 | 62099136 | 19.8997487 | 7.3434205 | 53.92582 | 2525253 |
| 397 | 157609 | 62570773 | 19.9248588 | 7.3495966 | 54.01657 | 2518892 |
| 398 | 158404 | 63044792 | 19.9499373 | 7.3557624 | 54.10724 | 2512563 |
| 399 | 159201 | 63521199 | 19.9749844 | 7.3619178 | 54.19783 | 2506266 |
| 400 | 160000 | 64000000 | 20.0000000 | 7.3680630 | 54.28835 | 2500000 |

## TABLE I.

| No. | Square | Cube | Square Root | Cube Root | Cube Root of the Square | Reciprocal 0.00 |
|---|---|---|---|---|---|---|
| 401 | 160801 | 64481201 | 20.0249844 | 7.3741979 | 54.37880 | 2493766 |
| 402 | 161604 | 64964808 | 20.0499377 | 7.3803227 | 54.46916 | 2487562 |
| 403 | 162409 | 65450827 | 20.0748599 | 7.3864373 | 54.55946 | 2481390 |
| 404 | 163216 | 65939264 | 20.0997512 | 7.3925418 | 54.64967 | 2475248 |
| 405 | 164025 | 66430125 | 20.1246118 | 7.3986363 | 54.73982 | 2469136 |
| 406 | 164836 | 66923416 | 20.1494417 | 7.4047206 | 54.82989 | 2463054 |
| 407 | 165649 | 67419143 | 20.1742410 | 7.4107950 | 54.91988 | 2457002 |
| 408 | 166464 | 67917312 | 20.1990099 | 7.4168595 | 55.00981 | 2450980 |
| 409 | 167281 | 68417929 | 20.2237484 | 7.4229142 | 55.09965 | 2444988 |
| 410 | 168100 | 68921000 | 20.2484567 | 7.4289589 | 55.18943 | 2439024 |
| 411 | 168921 | 69426531 | 20.2731349 | 7.4349938 | 55.27913 | 2433090 |
| 412 | 169744 | 69934528 | 20.2977831 | 7.4410189 | 55.36876 | 2427184 |
| 413 | 170569 | 70444997 | 20.3224014 | 7.4470342 | 55.45832 | 2421308 |
| 414 | 171396 | 70957944 | 20.3469899 | 7.4530399 | 55.54780 | 2415459 |
| 415 | 172225 | 71473375 | 20.3715488 | 7.4590359 | 55.63722 | 2409639 |
| 416 | 173056 | 71991296 | 20.3960781 | 7.4650223 | 55.72656 | 2403846 |
| 417 | 173889 | 72511713 | 20.4205779 | 7.4709991 | 55.81583 | 2398082 |
| 418 | 174724 | 73034632 | 20.4450483 | 7.4769664 | 55.90503 | 2392344 |
| 419 | 175561 | 73560059 | 20.4694895 | 7.4829242 | 55.99415 | 2386635 |
| 420 | 176400 | 74088000 | 20.4939015 | 7.4888724 | 56.08321 | 2380952 |
| 421 | 177241 | 74618461 | 20.5182845 | 7.4948113 | 56.17220 | 2375297 |
| 422 | 178084 | 75151448 | 20.5426386 | 7.5007406 | 56.26111 | 2369668 |
| 423 | 178929 | 75686967 | 20.5669638 | 7.5066607 | 56.34996 | 2364066 |
| 424 | 179776 | 76225024 | 20.5912603 | 7.5125715 | 56.43873 | 2358491 |
| 425 | 180625 | 76765625 | 20.6155281 | 7.5184730 | 56.52744 | 2352941 |
| 426 | 181476 | 77308776 | 20.6397674 | 7.5243652 | 56.61607 | 2347418 |
| 427 | 182329 | 77854483 | 20.6639783 | 7.5302482 | 56.70464 | 2341920 |
| 428 | 183184 | 78402752 | 20.6881609 | 7.5361221 | 56.79314 | 2336449 |
| 429 | 184041 | 78953589 | 20.7123152 | 7.5419867 | 56.88156 | 2331002 |
| 430 | 184900 | 79507000 | 20.7364414 | 7.5478423 | 56.96992 | 2325581 |
| 431 | 185761 | 80062991 | 20.7605395 | 7.5536888 | 57.05821 | 2320186 |
| 432 | 186624 | 80621568 | 20.7846097 | 7.5595263 | 57.14644 | 2314815 |
| 433 | 187489 | 81182737 | 20.8086520 | 7.5653548 | 57.23459 | 2309469 |
| 434 | 188356 | 81746504 | 20.8326667 | 7.5711743 | 57.32268 | 2304147 |
| 435 | 189225 | 82312875 | 20.8566536 | 7.5769849 | 57.41070 | 2298851 |
| 436 | 190096 | 82881856 | 20.8806130 | 7.5827865 | 57.49865 | 2293578 |
| 437 | 190969 | 83453453 | 20.9045450 | 7.5885793 | 57.58654 | 2288330 |
| 438 | 191844 | 84027672 | 20.9284495 | 7.5943633 | 57.67435 | 2283105 |
| 439 | 192721 | 84604519 | 20.9523268 | 7.6001385 | 57.76211 | 2277904 |
| 440 | 193600 | 85184000 | 20.9761770 | 7.6059049 | 57.84979 | 2272727 |
| 441 | 194481 | 85766121 | 21.0000000 | 7.6116626 | 57.93741 | 2267574 |
| 442 | 195364 | 86350888 | 21.0237960 | 7.6174116 | 58.02496 | 2262443 |
| 443 | 196249 | 86938307 | 21.0475652 | 7.6231519 | 58.11245 | 2257336 |
| 444 | 197136 | 87528384 | 21.0713075 | 7.6288837 | 58.19987 | 2252252 |
| 445 | 198025 | 88121125 | 21.0950231 | 7.6346067 | 58.28722 | 2247191 |
| 446 | 198916 | 88716536 | 21.1187121 | 7.6403213 | 58.37451 | 2242152 |
| 447 | 199809 | 89314623 | 21.1423745 | 7.6460272 | 58.46173 | 2237136 |
| 448 | 200704 | 89915392 | 21.1660105 | 7.6517247 | 58.54889 | 2232143 |
| 449 | 201601 | 90518849 | 21.1896201 | 7.6574138 | 58.63599 | 2227171 |
| 450 | 202500 | 91125000 | 21.2132034 | 7.6630943 | 58.72301 | 2222222 |

TABLE I.

| No. | Square | Cube | Square Root | Cube Root | Cube Root of the Square | Reciprocal 0.00 |
|---|---|---|---|---|---|---|
| 451 | 203401 | 91733851 | 21.2367606 | 7.6687665 | 58.80998 | 2217295 |
| 452 | 204304 | 92345408 | 21.2602916 | 7.6744303 | 58.89688 | 2212389 |
| 453 | 205209 | 92959677 | 21.2837967 | 7.6800857 | 58.98372 | 2207506 |
| 454 | 206116 | 93576664 | 21.3072758 | 7.6857328 | 59.07049 | 2202643 |
| 455 | 207025 | 94196375 | 21.3307290 | 7.6913717 | 59.15720 | 2197802 |
| 456 | 207936 | 94818816 | 21.3541565 | 7.6970023 | 59.24384 | 2192982 |
| 457 | 208849 | 95443993 | 21.3775583 | 7.7026246 | 59.33043 | 2188184 |
| 458 | 209764 | 96071912 | 21.4009346 | 7.7082388 | 59.41695 | 2183406 |
| 459 | 210681 | 96702579 | 21.4242853 | 7.7138448 | 59.50340 | 2178649 |
| 460 | 211600 | 97336000 | 21.4476106 | 7.7194426 | 59.58979 | 2173913 |
| 461 | 212521 | 97972181 | 21.4709106 | 7.7250325 | 59.67611 | 2169197 |
| 462 | 213444 | 98611128 | 21.4941853 | 7.7306141 | 59.76239 | 2164502 |
| 463 | 214369 | 99252847 | 21.5174348 | 7.7361877 | 59.84860 | 2159827 |
| 464 | 215296 | 99897344 | 21.5406592 | 7.7417532 | 59.93474 | 2155172 |
| 465 | 216225 | 100544625 | 21.5638587 | 7.7473109 | 60.02083 | 2150538 |
| 466 | 217156 | 101194696 | 21.5870331 | 7.7528606 | 60.10685 | 2145923 |
| 467 | 218089 | 101847563 | 21.6101828 | 7.7584023 | 60.19281 | 2141328 |
| 468 | 219024 | 102503232 | 21.6333077 | 7.7639361 | 60.27870 | 2136752 |
| 469 | 219961 | 103161709 | 21.6564078 | 7.7694620 | 60.36454 | 2132196 |
| 470 | 220900 | 103823000 | 21.6794834 | 7.7749801 | 60.45032 | 2127660 |
| 471 | 221841 | 104487111 | 21.7025344 | 7.7804904 | 60.53603 | 2123142 |
| 472 | 222784 | 105154048 | 21.7255610 | 7.7859928 | 60.62168 | 2118644 |
| 473 | 223729 | 105823817 | 21.7485632 | 7.7914875 | 60.70728 | 2114165 |
| 474 | 224676 | 106496424 | 21.7715411 | 7.7969745 | 60.79281 | 2109705 |
| 475 | 225625 | 107171875 | 21.7944947 | 7.8024538 | 60.87828 | 2105263 |
| 476 | 226576 | 107850176 | 21.8174242 | 7.8079254 | 60.96370 | 2100840 |
| 477 | 227529 | 108531333 | 21.8403297 | 7.8133892 | 61.04905 | 2096436 |
| 478 | 228484 | 109215352 | 21.8632111 | 7.8188456 | 61.13435 | 2092050 |
| 479 | 229441 | 109902239 | 21.8860686 | 7.8242942 | 61.21958 | 2087683 |
| 480 | 230400 | 110592000 | 21.9089023 | 7.8297353 | 61.30475 | 2083333 |
| 481 | 231361 | 111284641 | 21.9317122 | 7.8351688 | 61.38987 | 2079002 |
| 482 | 232324 | 111980168 | 21.9544984 | 7.8405949 | 61.47493 | 2074689 |
| 483 | 233289 | 112678587 | 21.9772610 | 7.8460134 | 61.55993 | 2070393 |
| 484 | 234256 | 113379904 | 22.0000000 | 7.8514244 | 61.64487 | 2066116 |
| 485 | 235225 | 114084125 | 22.0227155 | 7.8568281 | 61.72975 | 2061856 |
| 486 | 236196 | 114791256 | 22.0454077 | 7.8622242 | 61.81457 | 2057613 |
| 487 | 237169 | 115501303 | 22.0680765 | 7.8676130 | 61.89933 | 2053388 |
| 488 | 238144 | 116214272 | 22.0907270 | 7.8729944 | 61.98404 | 2049180 |
| 489 | 239121 | 116930169 | 22.1133444 | 7.8783684 | 62.06869 | 2044990 |
| 490 | 240100 | 117649000 | 22.1359436 | 7.8837352 | 62.15328 | 2040816 |
| 491 | 241081 | 118370771 | 22.1585198 | 7.8890946 | 62.23781 | 2036660 |
| 492 | 242064 | 119095488 | 22.1810730 | 7.8944468 | 62.32229 | 2032520 |
| 493 | 243049 | 119823157 | 22.2036033 | 7.8997917 | 62.40671 | 2028398 |
| 494 | 244036 | 120553784 | 22.2261108 | 7.9051294 | 62.49107 | 2024291 |
| 495 | 245025 | 121287375 | 22.2485955 | 7.9104599 | 62.57538 | 2020202 |
| 496 | 246016 | 122023936 | 22.2710575 | 7.9157832 | 62.65962 | 2016129 |
| 497 | 247009 | 122763473 | 22.2934968 | 7.9210994 | 62.74382 | 2012072 |
| 498 | 248004 | 123505992 | 22.3159136 | 7.9264085 | 62.82795 | 2008032 |
| 499 | 249001 | 124251499 | 22.3383079 | 7.9317104 | 62.91203 | 2004008 |
| 500 | 250000 | 125000000 | 22.3606798 | 7.9370053 | 62.99605 | 2000000 |

TABLE I.

| No. | Square | Cube | Square Root | Cube Root | Cube Root of the Square | Reciprocal 0.00 |
|---|---|---|---|---|---|---|
| 501 | 251001 | 125751501 | 22.3830293 | 7.9422931 | 63.08002 | 1996008 |
| 502 | 252004 | 126506008 | 22.4053565 | 7.9475739 | 63.16393 | 1992032 |
| 503 | 253009 | 127263527 | 22.4276615 | 7.9528477 | 63.24779 | 1988072 |
| 504 | 254016 | 128024064 | 22.4499443 | 7.9581144 | 63.33159 | 1984127 |
| 505 | 255025 | 128787625 | 22.4722051 | 7.9633743 | 63.41533 | 1980198 |
| 506 | 256036 | 129554216 | 22.4944438 | 7.9686271 | 63.49902 | 1976285 |
| 507 | 257049 | 130323843 | 22.5166605 | 7.9738731 | 63.58265 | 1972387 |
| 508 | 258064 | 131096512 | 22.5388553 | 7.9791122 | 63.66623 | 1968504 |
| 509 | 259081 | 131872229 | 22.5610283 | 7.9843444 | 63.74976 | 1964637 |
| 510 | 260100 | 132651000 | 22.5831796 | 7.9895697 | 63.83322 | 1960784 |
| 511 | 261121 | 133432831 | 22.6053091 | 7.9947883 | 63.91664 | 1956947 |
| 512 | 262144 | 134217728 | 22.6274170 | 8.0000000 | 64.00000 | 1953125 |
| 513 | 263169 | 135005697 | 22.6495033 | 8.0052049 | 64.08331 | 1949318 |
| 514 | 264196 | 135796744 | 22.6715681 | 8.0104032 | 64.16656 | 1945525 |
| 515 | 265225 | 136590875 | 22.6936114 | 8.0155946 | 64.24976 | 1941748 |
| 516 | 266256 | 137388096 | 22.7156334 | 8.0207794 | 64.33290 | 1937984 |
| 517 | 267289 | 138188413 | 22.7376340 | 8.0259574 | 64.41599 | 1934236 |
| 518 | 268324 | 138991832 | 22.7596134 | 8.0311287 | 64.49903 | 1930502 |
| 519 | 269361 | 139798359 | 22.7815715 | 8.0362935 | 64.58201 | 1926782 |
| 520 | 270400 | 140608000 | 22.8035085 | 8.0414515 | 64.66494 | 1923077 |
| 521 | 271441 | 141420761 | 22.8254244 | 8.0466030 | 64.74782 | 1919386 |
| 522 | 272484 | 142236648 | 22.8473193 | 8.0517479 | 64.83064 | 1915709 |
| 523 | 273529 | 143055667 | 22.8691933 | 8.0568862 | 64.91342 | 1912046 |
| 524 | 274576 | 143877824 | 22.8910463 | 8.0620180 | 64.99613 | 1908397 |
| 525 | 275625 | 144703125 | 22.9128785 | 8.0671432 | 65.07880 | 1904762 |
| 526 | 276676 | 145531576 | 22.9346899 | 8.0722620 | 65.16141 | 1901141 |
| 527 | 277729 | 146363183 | 22.9564806 | 8.0773743 | 65.24397 | 1897533 |
| 528 | 278784 | 147197952 | 22.9782506 | 8.0824800 | 65.32648 | 1893939 |
| 529 | 279841 | 148035889 | 23.0000000 | 8.0875794 | 65.40894 | 1890359 |
| 530 | 280900 | 148877000 | 23.0217289 | 8.0926723 | 65.49135 | 1886792 |
| 531 | 281961 | 149721291 | 23.0434372 | 8.0977589 | 65.57370 | 1883239 |
| 532 | 283024 | 150568768 | 23.0651252 | 8.1028390 | 65.65600 | 1879699 |
| 533 | 284089 | 151419437 | 23.0867928 | 8.1079128 | 65.73825 | 1876173 |
| 534 | 285156 | 152273304 | 23.1084400 | 8.1129803 | 65.82045 | 1872659 |
| 535 | 286225 | 153130375 | 23.1300670 | 8.1180414 | 65.90260 | 1869159 |
| 536 | 287296 | 153990656 | 23.1516738 | 8.1230962 | 65.98469 | 1865672 |
| 537 | 288369 | 154854153 | 23.1732605 | 8.1281447 | 66.06674 | 1862197 |
| 538 | 289444 | 155720872 | 23.1948270 | 8.1331870 | 66.14873 | 1858736 |
| 539 | 290521 | 156590819 | 23.2163735 | 8.1382230 | 66.23067 | 1855288 |
| 540 | 291600 | 157464000 | 23.2379001 | 8.1432529 | 66.31257 | 1851852 |
| 541 | 292681 | 158340421 | 23.2594067 | 8.1482765 | 66.39441 | 1848429 |
| 542 | 293764 | 159220088 | 23.2808935 | 8.1532939 | 66.47620 | 1845018 |
| 543 | 294849 | 160103007 | 23.3023604 | 8.1583051 | 66.55794 | 1841621 |
| 544 | 295936 | 160989184 | 23.3238076 | 8.1633102 | 66.63963 | 1838235 |
| 545 | 297025 | 161878625 | 23.3452351 | 8.1683092 | 66.72127 | 1834862 |
| 546 | 298116 | 162771336 | 23.3666429 | 8.1733020 | 66.80287 | 1831502 |
| 547 | 299209 | 163667323 | 23.3880311 | 8.1782888 | 66.88441 | 1828154 |
| 548 | 300304 | 164566592 | 23.4093998 | 8.1832695 | 66.96590 | 1824818 |
| 549 | 301401 | 165469149 | 23.4307490 | 8.1882441 | 67.04734 | 1821494 |
| 550 | 302500 | 166375000 | 23.4520788 | 8.1932127 | 67.12873 | 1818182 |

TABLE I.

| No. | Square | Cube | Square Root | Cube Root | Cube Root of the Square | Reciprocal 0.00 |
|---|---|---|---|---|---|---|
| 551 | 303601 | 167284151 | 23.4733892 | 8.1981753 | 67.21008 | 1814882 |
| 552 | 304704 | 168196608 | 23.4946802 | 8.2031319 | 67.29137 | 1811594 |
| 553 | 305809 | 169112377 | 23.5159520 | 8.2080825 | 67.37262 | 1808318 |
| 554 | 306916 | 170031464 | 23.5372046 | 8.2130271 | 67.45381 | 1805054 |
| 555 | 308025 | 170953875 | 23.5584380 | 8.2179657 | 67.53496 | 1801802 |
| 556 | 309136 | 171879616 | 23.5796522 | 8.2228985 | 67.61606 | 1798561 |
| 557 | 310249 | 172808693 | 23.6008474 | 8.2278254 | 67.69711 | 1795332 |
| 558 | 311364 | 173741112 | 23.6220236 | 8.2327463 | 67.77811 | 1792115 |
| 559 | 312481 | 174676879 | 23.6431808 | 8.2376614 | 67.85907 | 1788909 |
| 560 | 313600 | 175616000 | 23.6643191 | 8.2425706 | 67.93997 | 1785714 |
| 561 | 314721 | 176558481 | 23.6854386 | 8.2474740 | 68.02083 | 1782531 |
| 562 | 315844 | 177504328 | 23.7065392 | 8.2523715 | 68.10164 | 1779359 |
| 563 | 316969 | 178453547 | 23.7276210 | 8.2572633 | 68.18240 | 1776199 |
| 564 | 318096 | 179406144 | 23.7486842 | 8.2621492 | 68.26311 | 1773050 |
| 565 | 319225 | 180362125 | 23.7697286 | 8.2670294 | 68.34378 | 1769912 |
| 566 | 320356 | 181321496 | 23.7907545 | 8.2719039 | 68.42439 | 1766784 |
| 567 | 321489 | 182284263 | 23.8117618 | 8.2767726 | 68.50496 | 1763668 |
| 568 | 322624 | 183250432 | 23.8327506 | 8.2816355 | 68.58549 | 1760563 |
| 569 | 323761 | 184220009 | 23.8537209 | 8.2864928 | 68.66596 | 1757469 |
| 570 | 324900 | 185193000 | 23.8746728 | 8.2913444 | 68.74639 | 1754386 |
| 571 | 326041 | 186169411 | 23.8956063 | 8.2961903 | 68.82677 | 1751313 |
| 572 | 327184 | 187149248 | 23.9165215 | 8.3010304 | 68.90711 | 1748252 |
| 573 | 328329 | 188132517 | 23.9374184 | 8.3058651 | 68.98740 | 1745201 |
| 574 | 329476 | 189119224 | 23.9582971 | 8.3106941 | 69.06764 | 1742160 |
| 575 | 330625 | 190109375 | 23.9791576 | 8.3155175 | 69.14783 | 1739130 |
| 576 | 331776 | 191102976 | 24.0000000 | 8.3203353 | 69.22798 | 1736111 |
| 577 | 332929 | 192100033 | 24.0208243 | 8.3251475 | 69.30808 | 1733102 |
| 578 | 334084 | 193100552 | 24.0416306 | 8.3299542 | 69.38814 | 1730104 |
| 579 | 335241 | 194104539 | 24.0624188 | 8.3347553 | 69.46815 | 1727116 |
| 580 | 336400 | 195112000 | 24.0831891 | 8.3395509 | 69.54811 | 1724138 |
| 581 | 337561 | 196122941 | 24.1039416 | 8.3443410 | 69.62803 | 1721170 |
| 582 | 338724 | 197137368 | 24.1246762 | 8.3491256 | 69.70790 | 1718213 |
| 583 | 339889 | 198155287 | 24.1453929 | 8.3539047 | 69.78772 | 1715266 |
| 584 | 341056 | 199176704 | 24.1660919 | 8.3586784 | 69.86750 | 1712329 |
| 585 | 342225 | 200201625 | 24.1867732 | 8.3634466 | 69.94724 | 1709402 |
| 586 | 343396 | 201230056 | 24.2074369 | 8.3682095 | 70.02693 | 1706485 |
| 587 | 344569 | 202262003 | 24.2280829 | 8.3729668 | 70.10657 | 1703578 |
| 588 | 345744 | 203297472 | 24.2487113 | 8.3777188 | 70.18617 | 1700680 |
| 589 | 346921 | 204336469 | 24.2693222 | 8.3824653 | 70.26572 | 1697793 |
| 590 | 348100 | 205379000 | 24.2899156 | 8.3872065 | 70.34523 | 1694915 |
| 591 | 349281 | 206425071 | 24.3104916 | 8.3919423 | 70.42470 | 1692047 |
| 592 | 350464 | 207474688 | 24.3310501 | 8.3966729 | 70.50412 | 1689189 |
| 593 | 351649 | 208527857 | 24.3515913 | 8.4013981 | 70.58349 | 1686341 |
| 594 | 352836 | 209584584 | 24.3721152 | 8.4061180 | 70.66282 | 1683502 |
| 595 | 354025 | 210644875 | 24.3926218 | 8.4108326 | 70.74210 | 1680672 |
| 596 | 355216 | 211708736 | 24.4131112 | 8.4155419 | 70.82135 | 1677852 |
| 597 | 356409 | 212776173 | 24.4335834 | 8.4202460 | 70.90054 | 1675042 |
| 598 | 357604 | 213847192 | 24.4540385 | 8.4249448 | 70.97969 | 1672241 |
| 599 | 358801 | 214921799 | 24.4744765 | 8.4296383 | 71.05880 | 1669449 |
| 600 | 360000 | 216000000 | 24.4948974 | 8.4343267 | 71.13787 | 1666667 |

TABLE I.

| No. | Square | Cube | Square Root | Cube Root | Cube Root of the Square | Reciprocal 0.00 |
|---|---|---|---|---|---|---|
| 601 | 361201 | 217081801 | 24.5153013 | 8.4390098 | 71.21689 | 1663894 |
| 602 | 362404 | 218167208 | 24.5356883 | 8.4436877 | 71.29586 | 1661130 |
| 603 | 363609 | 219256227 | 24.5560583 | 8.4483605 | 71.37480 | 1658375 |
| 604 | 364816 | 220348864 | 24.5764115 | 8.4530281 | 71.45368 | 1655629 |
| 605 | 366025 | 221445125 | 24.5967478 | 8.4576906 | 71.53253 | 1652893 |
| 606 | 367236 | 222545016 | 24.6170673 | 8.4623479 | 71.61133 | 1650165 |
| 607 | 368449 | 223648543 | 24.6373700 | 8.4670000 | 71.69009 | 1647446 |
| 608 | 369664 | 224755712 | 24.6576560 | 8.4716471 | 71.76881 | 1644737 |
| 609 | 370881 | 225866529 | 24.6779254 | 8.4762892 | 71.84748 | 1642036 |
| 610 | 372100 | 226981000 | 24.6981781 | 8.4809261 | 71.92611 | 1639344 |
| 611 | 373321 | 228099131 | 24.7184142 | 8.4855579 | 72.00469 | 1636661 |
| 612 | 374544 | 229220928 | 24.7386338 | 8.4901848 | 72.08324 | 1633987 |
| 613 | 375769 | 230346397 | 24.7588368 | 8.4948065 | 72.16174 | 1631321 |
| 614 | 376996 | 231475544 | 24.7790234 | 8.4994233 | 72.24020 | 1628664 |
| 615 | 378225 | 232608375 | 24.7991935 | 8.5040350 | 72.31861 | 1626016 |
| 616 | 379456 | 233744896 | 24.8193473 | 8.5086417 | 72.39698 | 1623377 |
| 617 | 380689 | 234885113 | 24.8394847 | 8.5132435 | 72.47531 | 1620746 |
| 618 | 381924 | 236029032 | 24.8596058 | 8.5178403 | 72.55360 | 1618123 |
| 619 | 383161 | 237176659 | 24.8797106 | 8.5224321 | 72.63185 | 1615509 |
| 620 | 384400 | 238328000 | 24.8997992 | 8.5270189 | 72.71005 | 1612903 |
| 621 | 385641 | 239483061 | 24.9198716 | 8.5316009 | 72.78821 | 1610306 |
| 622 | 386884 | 240641848 | 24.9399278 | 8.5361780 | 72.86633 | 1607717 |
| 623 | 388129 | 241804367 | 24.9599679 | 8.5407501 | 72.94441 | 1605136 |
| 624 | 389376 | 242970624 | 24.9799920 | 8.5453173 | 73.02245 | 1602564 |
| 625 | 390625 | 244140625 | 25.0000000 | 8.5498797 | 73.10044 | 1600000 |
| 626 | 391876 | 245314376 | 25.0199920 | 8.5544372 | 73.17840 | 1597444 |
| 627 | 393129 | 246491883 | 25.0399681 | 8.5589899 | 73.25631 | 1594896 |
| 628 | 394384 | 247673152 | 25.0599282 | 8.5635377 | 73.33418 | 1592357 |
| 629 | 395641 | 248858189 | 25.0798724 | 8.5680807 | 73.41201 | 1589825 |
| 630 | 396900 | 250047000 | 25.0998008 | 8.5726189 | 73.48979 | 1587302 |
| 631 | 398161 | 251239591 | 25.1197134 | 8.5771523 | 73.56754 | 1584786 |
| 632 | 399424 | 252435968 | 25.1396102 | 8.5816809 | 73.64525 | 1582278 |
| 633 | 400689 | 253636137 | 25.1594913 | 8.5862047 | 73.72291 | 1579779 |
| 634 | 401956 | 254840104 | 25.1793566 | 8.5907238 | 73.80053 | 1577287 |
| 635 | 403225 | 256047875 | 25.1992063 | 8.5952380 | 73.87812 | 1574803 |
| 636 | 404496 | 257259456 | 25.2190404 | 8.5997476 | 73.95566 | 1572327 |
| 637 | 405769 | 258474853 | 25.2388589 | 8.6042525 | 74.03316 | 1569859 |
| 638 | 407044 | 259694072 | 25.2586619 | 8.6087526 | 74.11062 | 1567398 |
| 639 | 408321 | 260917119 | 25.2784493 | 8.6132480 | 74.18804 | 1564945 |
| 640 | 409600 | 262144000 | 25.2982213 | 8.6177388 | 74.26542 | 1562500 |
| 641 | 410881 | 263374721 | 25.3179778 | 8.6222248 | 74.34276 | 1560062 |
| 642 | 412164 | 264609288 | 25.3377189 | 8.6267063 | 74.42006 | 1557632 |
| 643 | 413449 | 265847707 | 25.3574447 | 8.6311830 | 74.49732 | 1555210 |
| 644 | 414736 | 267089984 | 25.3771551 | 8.6356551 | 74.57454 | 1552795 |
| 645 | 416025 | 268336125 | 25.3968502 | 8.6401226 | 74.65172 | 1550388 |
| 646 | 417316 | 269586136 | 25.4165301 | 8.6445855 | 74.72886 | 1547988 |
| 647 | 418609 | 270840023 | 25.4361947 | 8.6490437 | 74.80596 | 1545595 |
| 648 | 419904 | 272097792 | 25.4558441 | 8.6534974 | 74.88302 | 1543210 |
| 649 | 421201 | 273359449 | 25.4754784 | 8.6579465 | 74.96004 | 1540832 |
| 650 | 422500 | 274625000 | 25.4950976 | 8.6623911 | 75.03702 | 1538462 |

| No. | Square | Cube | Square Root | Cube Root | Cube Root of the Square | Reciprocal 0.00 |
|---|---|---|---|---|---|---|
| 651 | 423801 | 275894451 | 25.5147016 | 8.6668310 | 75.11396 | 1536098 |
| 652 | 425104 | 277167808 | 25.5342907 | 8.6712665 | 75.19086 | 1533742 |
| 653 | 426409 | 278445077 | 25.5538647 | 8.6756974 | 75.26772 | 1531394 |
| 654 | 427716 | 279726264 | 25.5734237 | 8.6801237 | 75.34455 | 1529052 |
| 655 | 429025 | 281011375 | 25.5929678 | 8.6845456 | 75.42133 | 1526718 |
| 656 | 430336 | 282300416 | 25.6124969 | 8.6889630 | 75.49808 | 1524390 |
| 657 | 431649 | 283593393 | 25.6320112 | 8.6933759 | 75.57478 | 1522070 |
| 658 | 432964 | 284890312 | 25.6515107 | 8.6977843 | 75.65145 | 1519757 |
| 659 | 434281 | 286191179 | 25.6709953 | 8.7021882 | 75.72808 | 1517451 |
| 660 | 435600 | 287496000 | 25.6904652 | 8.7065877 | 75.80467 | 1515152 |
| 661 | 436921 | 288804781 | 25.7099203 | 8.7109827 | 75.88122 | 1512859 |
| 662 | 438244 | 290117528 | 25.7293607 | 8.7153734 | 75.95773 | 1510574 |
| 663 | 439569 | 291434247 | 25.7487864 | 8.7197596 | 76.03421 | 1508296 |
| 664 | 440896 | 292754944 | 25.7681975 | 8.7241414 | 76.11064 | 1506024 |
| 665 | 442225 | 294079625 | 25.7875939 | 8.7285187 | 76.18704 | 1503759 |
| 666 | 443556 | 295408296 | 25.8069758 | 8.7328918 | 76.26340 | 1501502 |
| 667 | 444889 | 296740963 | 25.8263431 | 8.7372604 | 76.33972 | 1499250 |
| 668 | 446224 | 298077632 | 25.8456960 | 8.7416246 | 76.41600 | 1497006 |
| 669 | 447561 | 299418309 | 25.8650343 | 8.7459846 | 76.49225 | 1494768 |
| 670 | 448900 | 300763000 | 25.8843582 | 8.7503401 | 76.56845 | 1492537 |
| 671 | 450241 | 302111711 | 25.9036677 | 8.7546913 | 76.64462 | 1490313 |
| 672 | 451584 | 303464448 | 25.9229628 | 8.7590383 | 76.72075 | 1488095 |
| 673 | 452929 | 304821217 | 25.9422435 | 8.7633809 | 76.79684 | 1485884 |
| 674 | 454276 | 306182024 | 25.9615100 | 8.7677192 | 76.87290 | 1483680 |
| 675 | 455625 | 307546875 | 25.9807621 | 8.7720532 | 76.94892 | 1481481 |
| 676 | 456976 | 308915776 | 26.0000000 | 8.7763830 | 77.02490 | 1479290 |
| 677 | 458329 | 310288733 | 26.0192237 | 8.7807084 | 77.10084 | 1477105 |
| 678 | 459684 | 311665752 | 26.0384331 | 8.7850296 | 77.17675 | 1474926 |
| 679 | 461041 | 313046839 | 26.0576284 | 8.7893466 | 77.25261 | 1472754 |
| 680 | 462400 | 314432000 | 26.0768096 | 8.7936593 | 77.32844 | 1470588 |
| 681 | 463761 | 315821241 | 26.0959767 | 8.7979679 | 77.40424 | 1468429 |
| 682 | 465124 | 317214568 | 26.1151297 | 8.8022721 | 77.47999 | 1466276 |
| 683 | 466489 | 318611987 | 26.1342687 | 8.8065722 | 77.55571 | 1464129 |
| 684 | 467856 | 320013504 | 26.1533937 | 8.8108681 | 77.63140 | 1461988 |
| 685 | 469225 | 321419125 | 26.1725047 | 8.8151598 | 77.70704 | 1459854 |
| 686 | 470596 | 322828856 | 26.1916017 | 8.8194474 | 77.78265 | 1457726 |
| 687 | 471969 | 324242703 | 26.2106848 | 8.8237307 | 77.85822 | 1455604 |
| 688 | 473344 | 325660672 | 26.2297541 | 8.8280099 | 77.93376 | 1453488 |
| 689 | 474721 | 327082769 | 26.2488095 | 8.8322850 | 78.00926 | 1451379 |
| 690 | 476100 | 328509000 | 26.2678511 | 8.8365559 | 78.08472 | 1449275 |
| 691 | 477481 | 329939371 | 26.2868789 | 8.8408227 | 78.16015 | 1447178 |
| 692 | 478864 | 331373888 | 26.3058929 | 8.8450854 | 78.23554 | 1445087 |
| 693 | 480249 | 332812557 | 26.3248932 | 8.8493440 | 78.31089 | 1443001 |
| 694 | 481636 | 334255384 | 26.3438797 | 8.8535985 | 78.38621 | 1440922 |
| 695 | 483025 | 335702375 | 26.3628527 | 8.8578489 | 78.46149 | 1438849 |
| 696 | 484416 | 337153536 | 26.3818119 | 8.8620952 | 78.53673 | 1436782 |
| 679 | 485809 | 338608873 | 26.4007576 | 8.8663375 | 78.61194 | 1434720 |
| 698 | 487204 | 340068392 | 26.4196896 | 8.8705757 | 78.68711 | 1432665 |
| 699 | 488601 | 341532099 | 26.4386081 | 8.8748099 | 78.76225 | 1430615 |
| 700 | 490000 | 343000000 | 26.4575131 | 8.8790400 | 78.83735 | 1428571 |

TABLE I.

| No. | Square | Cube | Square Root | Cube Root | Cube Root of the Square | Reciprocal 0.00 |
|---|---|---|---|---|---|---|
| 701 | 491401 | 344472101 | 26.4764046 | 8.8832661 | 78.91242 | 1426534 |
| 702 | 492804 | 345948408 | 26.4952826 | 8.8874882 | 78.98745 | 1424501 |
| 703 | 494209 | 347428927 | 26.5141472 | 8.8917063 | 79.06244 | 1422475 |
| 704 | 495616 | 348913664 | 26.5329983 | 8.8959204 | 79.13740 | 1420455 |
| 705 | 497025 | 350402625 | 26.5518361 | 8.9001304 | 79.21232 | 1418440 |
| 706 | 498436 | 351895816 | 26.5706605 | 8.9043366 | 79.28721 | 1416431 |
| 707 | 499849 | 353393243 | 26.5894716 | 8.9085387 | 79.36206 | 1414427 |
| 708 | 501264 | 354894912 | 26.6082694 | 8.9127369 | 79.43688 | 1412429 |
| 709 | 502681 | 356400829 | 26.6270539 | 8.9169311 | 79.51166 | 1410437 |
| 710 | 504100 | 357911000 | 26.6458252 | 8.9211214 | 79.58641 | 1408451 |
| 711 | 505521 | 359425431 | 26.6645833 | 8.9253078 | 79.66112 | 1406470 |
| 712 | 506944 | 360944128 | 26.6833281 | 8.9294902 | 79.73580 | 1404494 |
| 713 | 508369 | 362467097 | 26.7020598 | 8.9336687 | 79.81044 | 1402525 |
| 714 | 509796 | 363994344 | 26.7207784 | 8.9378433 | 79.88504 | 1400560 |
| 715 | 511225 | 365525875 | 26.7394839 | 8.9420140 | 79.95962 | 1398601 |
| 716 | 512656 | 367061696 | 26.7581763 | 8.9461809 | 80.03415 | 1396648 |
| 717 | 514089 | 368601813 | 26.7768557 | 8.9503438 | 80.10865 | 1394700 |
| 718 | 515524 | 370146232 | 26.7955220 | 8.9545029 | 80.18312 | 1392758 |
| 719 | 516961 | 371694959 | 26.8141754 | 8.9586581 | 80.25756 | 1390821 |
| 720 | 518400 | 373248000 | 26.8328157 | 8.9628095 | 80.33195 | 1388889 |
| 721 | 519841 | 374805361 | 26.8514432 | 8.9669570 | 80.40632 | 1386963 |
| 722 | 521284 | 376367048 | 26.8700577 | 8.9711007 | 80.48065 | 1385042 |
| 723 | 522729 | 377933067 | 26.8886593 | 8.9752406 | 80.55494 | 1383126 |
| 724 | 524176 | 379503424 | 26.9072481 | 8.9793766 | 80.62920 | 1381215 |
| 725 | 525625 | 381078125 | 26.9258240 | 8.9835089 | 80.70343 | 1379310 |
| 726 | 527076 | 382657176 | 26.9443872 | 8.9876373 | 80.77763 | 1377410 |
| 727 | 528529 | 384240583 | 26.9629375 | 8.9917620 | 80.85178 | 1375516 |
| 728 | 529984 | 385828352 | 26.9814751 | 8.9958829 | 80.92591 | 1373626 |
| 729 | 531441 | 387420489 | 27.0000000 | 9.0000000 | 81.00000 | 1371742 |
| 730 | 532900 | 389017000 | 27.0185122 | 9.0041134 | 81.07406 | 1369863 |
| 731 | 534361 | 390617891 | 27.0370117 | 9.0082229 | 81.14808 | 1367989 |
| 732 | 535824 | 392223168 | 27.0554985 | 9.0123288 | 81.22207 | 1366120 |
| 733 | 537289 | 393832837 | 27.0739727 | 9.0164309 | 81.29603 | 1364256 |
| 734 | 538756 | 395446904 | 27.0924344 | 9.0205293 | 81.36995 | 1362398 |
| 735 | 540225 | 397065375 | 27.1108834 | 9.0246239 | 81.44384 | 1360544 |
| 736 | 541696 | 398688256 | 27.1293199 | 9.0287149 | 81.51769 | 1358696 |
| 737 | 543169 | 400315553 | 27.1477439 | 9.0328021 | 81.59151 | 1356852 |
| 738 | 544644 | 401947272 | 27.1661554 | 9.0368857 | 81.66530 | 1355014 |
| 739 | 546121 | 403583419 | 27.1845544 | 9.0409655 | 81.73906 | 1353180 |
| 740 | 547600 | 405224000 | 27.2029410 | 9.0450417 | 81.81278 | 1351351 |
| 741 | 549081 | 406869021 | 27.2213152 | 9.0491142 | 81.88647 | 1349528 |
| 742 | 550564 | 408518488 | 27.2396769 | 9.0531831 | 81.96012 | 1347709 |
| 743 | 552049 | 410172407 | 27.2580263 | 9.0572482 | 82.03375 | 1345895 |
| 744 | 553536 | 411830784 | 27.2763634 | 9.0613098 | 82.10734 | 1344086 |
| 745 | 555025 | 413493625 | 27.2946881 | 9.0653677 | 82.18089 | 1342282 |
| 746 | 556516 | 415160936 | 27.3130006 | 9.0694220 | 82.25442 | 1340483 |
| 747 | 558009 | 416832723 | 27.3313007 | 9.0734726 | 82.32791 | 1338688 |
| 748 | 559504 | 418508992 | 27.3495887 | 9.0775197 | 82.40136 | 1336898 |
| 749 | 561001 | 420189749 | 27.3678644 | 9.0815631 | 82.47479 | 1335113 |
| 750 | 562500 | 421875000 | 27.3861279 | 9.0856030 | 82.54818 | 1333333 |

TABLE I.

| No. | Square | Cube | Square Root | Cube Root | Cube Root of the Square | Reciprocal 0.00 |
|---|---|---|---|---|---|---|
| 751 | 564001 | 423564751 | 27.4043792 | 9.0896392 | 82.62154 | 1331558 |
| 752 | 565504 | 425259008 | 27.4226184 | 9.0936719 | 82.69487 | 1329787 |
| 753 | 567009 | 426957777 | 27.4408455 | 9.0977010 | 82.76816 | 1328021 |
| 754 | 568516 | 428661064 | 27.4590604 | 9.1017265 | 82.84143 | 1326260 |
| 755 | 570025 | 430368875 | 27.4772633 | 9.1057485 | 82.91466 | 1324503 |
| 756 | 571536 | 432081216 | 27.4954542 | 9.1097669 | 82.98785 | 1322751 |
| 757 | 573049 | 433798093 | 27.5136330 | 9.1137818 | 83.06102 | 1321004 |
| 758 | 574564 | 435519512 | 27.5317998 | 9.1177931 | 83.13415 | 1319261 |
| 759 | 576081 | 437245479 | 27.5499546 | 9.1218010 | 83.20725 | 1317523 |
| 760 | 577600 | 438976000 | 27.5680975 | 9.1258053 | 83.28032 | 1315789 |
| 761 | 579121 | 440711081 | 27.5862284 | 9.1298061 | 83.35336 | 1314060 |
| 762 | 580644 | 442450728 | 27.6043475 | 9.1338034 | 83.42636 | 1312336 |
| 763 | 582169 | 444194947 | 27.6224546 | 9.1377971 | 83.49934 | 1310616 |
| 764 | 583696 | 445943744 | 27.6405499 | 9.1417874 | 83.57228 | 1308901 |
| 765 | 585225 | 447697125 | 27.6586334 | 9.1457742 | 83.64519 | 1307190 |
| 766 | 586756 | 449455096 | 27.6767050 | 9.1497576 | 83.71806 | 1305483 |
| 767 | 588289 | 451217663 | 27.6947648 | 9.1537375 | 83.79091 | 1303781 |
| 768 | 589824 | 452984832 | 27.7128129 | 9.1577139 | 83.86372 | 1302083 |
| 769 | 591361 | 454756609 | 27.7308492 | 9.1616869 | 83.93651 | 1300390 |
| 770 | 592900 | 456533000 | 27.7488739 | 9.1656565 | 84.00926 | 1298701 |
| 771 | 594441 | 458314011 | 27.7668868 | 9.1696225 | 84.08198 | 1297017 |
| 772 | 595984 | 460099648 | 27.7848880 | 9.1735852 | 84.15467 | 1295337 |
| 773 | 597529 | 461889917 | 27.8028775 | 9.1775445 | 84.22732 | 1293661 |
| 774 | 599076 | 463684824 | 27.8208555 | 9.1815003 | 84.29995 | 1291990 |
| 775 | 600625 | 465484375 | 27.8388218 | 9.1854527 | 84.37254 | 1290323 |
| 776 | 602176 | 467288576 | 27.8567766 | 9.1894018 | 84.44511 | 1288660 |
| 777 | 603729 | 469097433 | 27.8747197 | 9.1933474 | 84.51764 | 1287001 |
| 778 | 605284 | 470910952 | 27.8926514 | 9.1972897 | 84.59014 | 1285347 |
| 779 | 606841 | 472729139 | 27.9105715 | 9.2012286 | 84.66261 | 1283697 |
| 780 | 608400 | 474552000 | 27.9284801 | 9.2051641 | 84.73505 | 1282051 |
| 781 | 609961 | 476379541 | 27.9463772 | 9.2090962 | 84.80745 | 1280410 |
| 782 | 611524 | 478211768 | 27.9642629 | 9.2130250 | 84.87983 | 1278772 |
| 783 | 613089 | 480048687 | 27.9821372 | 9.2169505 | 84.95218 | 1277139 |
| 784 | 614656 | 481890304 | 28.0000000 | 9.2208726 | 85.02449 | 1275510 |
| 785 | 616225 | 483736625 | 28.0178515 | 9.2247914 | 85.09678 | 1273885 |
| 786 | 617796 | 485587656 | 28.0356915 | 9.2287068 | 85.16903 | 1272265 |
| 787 | 619369 | 487443403 | 28.0535203 | 9.2326189 | 85.24125 | 1270648 |
| 788 | 620944 | 489303872 | 28.0713377 | 9.2365277 | 85.31344 | 1269036 |
| 789 | 622521 | 491169069 | 28.0891438 | 9.2404333 | 85.38561 | 1267427 |
| 790 | 624100 | 493039000 | 28.1069386 | 9.2443355 | 85.45774 | 1265823 |
| 791 | 625681 | 494913671 | 28.1247222 | 9.2482344 | 85.52984 | 1264223 |
| 792 | 627264 | 496793088 | 28.1424946 | 9.2521300 | 85.60191 | 1262626 |
| 793 | 628849 | 498677257 | 28.1602557 | 9.2560224 | 85.67395 | 1261034 |
| 794 | 630436 | 500566184 | 28.1780056 | 9.2599114 | 85.74596 | 1259446 |
| 795 | 632025 | 502459875 | 28.1957444 | 9.2637973 | 85.81794 | 1257862 |
| 796 | 633616 | 504358336 | 28.2134720 | 9.2676798 | 85.88989 | 1256281 |
| 797 | 635209 | 506261573 | 28.2311884 | 9.2715592 | 85.96181 | 1254705 |
| 798 | 636804 | 508169592 | 28.2488938 | 9.2754352 | 86.03370 | 1253133 |
| 799 | 638401 | 510082399 | 28.2665881 | 9.2793081 | 86.10556 | 1251564 |
| 800 | 640000 | 512000000 | 28.2842712 | 9.2831777 | 86.17739 | 1250000 |

## TABLE I.

| No. | Square | Cube | Square Root | Cube Root | Cube Root of the Square | Reciprocal 0.00 |
|---|---|---|---|---|---|---|
| **801** | 641601 | 513922401 | 28.3019434 | 9.2870440 | 86.24919 | 1248439 |
| **802** | 643204 | 515849608 | 28.3196045 | 9.2909072 | 86.32096 | 1246883 |
| **803** | 644809 | 517781627 | 28.3372546 | 9.2947671 | 86.39270 | 1245330 |
| **804** | 646416 | 519718464 | 28.3548938 | 9.2986239 | 86.46441 | 1243781 |
| **805** | 648025 | 521660125 | 28.3725219 | 9.3024775 | 86.53609 | 1242236 |
| **806** | 649636 | 523606616 | 28.3901391 | 9.3063278 | 86.60774 | 1240695 |
| **807** | 651249 | 525557943 | 28.4077454 | 9.3101750 | 86.67936 | 1239157 |
| **808** | 652864 | 527514112 | 28.4253408 | 9.3140190 | 86.75095 | 1237624 |
| **809** | 654481 | 529475129 | 28.4429253 | 9.3178599 | 86.82251 | 1236094 |
| **810** | 656100 | 531441000 | 28.4604989 | 9.3216975 | 86.89404 | 1234568 |
| **811** | 657721 | 533411731 | 28.4780617 | 9.3255320 | 86.96555 | 1233046 |
| **812** | 659344 | 535387328 | 28.4956137 | 9.3293634 | 87.03702 | 1231527 |
| **813** | 660969 | 537367797 | 28.5131549 | 9.3331916 | 87.10847 | 1230012 |
| **814** | 662596 | 539353144 | 28.5306852 | 9.3370167 | 87.17988 | 1228501 |
| **815** | 664225 | 541343375 | 28.5482048 | 9.3408386 | 87.25127 | 1226994 |
| **816** | 665856 | 543338496 | 28.5657137 | 9.3446575 | 87.32262 | 1225490 |
| **817** | 667489 | 545338513 | 28.5832119 | 9.3484731 | 87.39395 | 1223990 |
| **818** | 669124 | 547343432 | 28.6006993 | 9.3522857 | 87.46525 | 1222494 |
| **819** | 670761 | 549353259 | 28.6181760 | 9.3560952 | 87.53652 | 1221001 |
| **820** | 672400 | 551368000 | 28.6356421 | 9.3599016 | 87.60776 | 1219512 |
| **821** | 674041 | 553387661 | 28.6530976 | 9.3637049 | 87.67897 | 1218027 |
| **822** | 675684 | 555412248 | 28.6705424 | 9.3675051 | 87.75015 | 1216545 |
| **823** | 677329 | 557441767 | 28.6879766 | 9.3713022 | 87.82131 | 1215067 |
| **824** | 678976 | 559476224 | 28.7054002 | 9.3750963 | 87.89243 | 1213592 |
| **825** | 680625 | 561515625 | 28.7228132 | 9.3788873 | 87.96353 | 1212121 |
| **826** | 682276 | 563559976 | 28.7402157 | 9.3826752 | 88.03459 | 1210654 |
| **827** | 683929 | 565609283 | 28.7576077 | 9.3864600 | 88.10563 | 1209190 |
| **828** | 685584 | 567663552 | 28.7749891 | 9.3902419 | 88.17664 | 1207729 |
| **829** | 687241 | 569722789 | 28.7923601 | 9.3940206 | 88.24762 | 1206273 |
| **830** | 688900 | 571787000 | 28.8097206 | 9.3977964 | 88.31858 | 1204819 |
| **831** | 690561 | 573856191 | 28.8270706 | 9.4015691 | 88.38950 | 1203369 |
| **832** | 692224 | 575930368 | 28.8444102 | 9.4053387 | 88.46040 | 1201923 |
| **833** | 693889 | 578009537 | 28.8617394 | 9.4091054 | 88.53126 | 1200480 |
| **834** | 695556 | 580093704 | 28.8790582 | 9.4128690 | 88.60210 | 1199041 |
| **835** | 697225 | 582182875 | 28.8963666 | 9.4166297 | 88.67291 | 1197605 |
| **836** | 698896 | 584277056 | 28.9136646 | 9.4203873 | 88.74370 | 1196172 |
| **837** | 700569 | 586376253 | 28.9309523 | 9.4241420 | 88.81445 | 1194743 |
| **838** | 702244 | 588480472 | 28.9482297 | 9.4278936 | 88.88518 | 1193317 |
| **839** | 703921 | 590589719 | 28.9654967 | 9.4316423 | 88.95588 | 1191895 |
| **840** | 705600 | 592704000 | 28.9827535 | 9.4353880 | 89.02655 | 1190476 |
| **841** | 707281 | 594823321 | 29.0000000 | 9.4391307 | 89.09719 | 1189061 |
| **842** | 708964 | 596947688 | 29.0172363 | 9.4428704 | 89.16780 | 1187648 |
| **843** | 710649 | 599077107 | 29.0344623 | 9.4466072 | 89.23839 | 1186240 |
| **844** | 712336 | 601211584 | 29.0516781 | 9.4503410 | 89.30895 | 1184834 |
| **845** | 714025 | 603351125 | 29.0688837 | 9.4540719 | 89.37948 | 1183432 |
| **846** | 715716 | 605495736 | 29.0860791 | 9.4577999 | 89.44998 | 1182033 |
| **847** | 717409 | 607645423 | 29.1032644 | 9.4615249 | 89.52045 | 1180638 |
| **848** | 719104 | 609800192 | 29.1204396 | 9.4652470 | 89.59090 | 1179245 |
| **849** | 720801 | 611960049 | 29.1376046 | 9.4689661 | 89.66132 | 1177856 |
| **850** | 722500 | 614125000 | 29.1547595 | 9.4726824 | 89.73171 | 1176471 |

TABLE I.

| No. | Square | Cube | Square Root | Cube Root | Cube Root of the Square | Reciprocal 0.00 |
|---|---|---|---|---|---|---|
| **851** | 724201 | 616295051 | 29.1719043 | 9.4763957 | 89.80208 | 1175088 |
| **852** | 725904 | 618470208 | 29.1890390 | 9.4801061 | 89.87241 | 1173709 |
| **853** | 727609 | 620650477 | 29.2061637 | 9.4838136 | 89.94272 | 1172333 |
| **854** | 729316 | 622835864 | 29.2232784 | 9.4875182 | 90.01300 | 1170960 |
| **855** | 731025 | 625026375 | 29.2403830 | 9.4912200 | 90.08326 | 1169591 |
| **856** | 732736 | 627222016 | 29.2574777 | 9.4949188 | 90.15348 | 1168224 |
| **857** | 734449 | 629422793 | 29.2745623 | 9.4986147 | 90.22368 | 1166861 |
| **858** | 736164 | 631628712 | 29.2916370 | 9.5023078 | 90.29385 | 1165501 |
| **859** | 737881 | 633839779 | 29.3087018 | 9.5059980 | 90.36400 | 1164144 |
| **860** | 739600 | 636056000 | 29.3257566 | 9.5096854 | 90.43412 | 1162791 |
| **861** | 741321 | 638277381 | 29.3428015 | 9.5133699 | 90.50421 | 1161440 |
| **862** | 743044 | 640503928 | 29.3598365 | 9.5170515 | 90.57427 | 1160093 |
| **863** | 744769 | 642735647 | 29.3768616 | 9.5207303 | 90.64431 | 1158749 |
| **864** | 746496 | 644972544 | 29.3938769 | 9.5244063 | 90.71432 | 1157407 |
| **865** | 748225 | 647214625 | 29.4108823 | 9.5280794 | 90.78430 | 1156069 |
| **866** | 749956 | 649461896 | 29.4278779 | 9.5317497 | 90.85425 | 1154734 |
| **867** | 751689 | 651714363 | 29.4448637 | 9.5354172 | 90.92418 | 1153403 |
| **868** | 753424 | 653972032 | 29.4618397 | 9.5390818 | 90.99408 | 1152074 |
| **869** | 755161 | 656234909 | 29.4788059 | 9.5427437 | 91.06396 | 1150748 |
| **870** | 756900 | 658503000 | 29.4957624 | 9.5464027 | 91.13380 | 1149425 |
| **871** | 758641 | 660776311 | 29.5127091 | 9.5500589 | 91.20363 | 1148106 |
| **872** | 760384 | 663054848 | 29.5296461 | 9.5537123 | 91.27342 | 1146789 |
| **873** | 762129 | 665338617 | 29.5465734 | 9.5573630 | 91.34319 | 1145475 |
| **874** | 763876 | 667627624 | 29.5634910 | 9.5610108 | 91.41293 | 1144165 |
| **875** | 765625 | 669921875 | 29.5803989 | 9.5646559 | 91.48264 | 1142857 |
| **876** | 767376 | 672221376 | 29.5972972 | 9.5682982 | 91.55233 | 1141553 |
| **877** | 769129 | 674526133 | 29.6141858 | 9.5719377 | 91.62199 | 1140251 |
| **878** | 770884 | 676836152 | 29.6310648 | 9.5755745 | 91.69163 | 1138952 |
| **879** | 772641 | 679151439 | 29.6479342 | 9.5792085 | 91.76124 | 1137656 |
| **880** | 774400 | 681472000 | 29.6647939 | 9.5828397 | 91.83082 | 1136364 |
| **881** | 776161 | 683797841 | 29.6816442 | 9.5864682 | 91.90037 | 1135074 |
| **882** | 777924 | 686128968 | 29.6984848 | 9.5900939 | 91.96990 | 1133787 |
| **883** | 779689 | 688465387 | 29.7153159 | 9.5937169 | 92.03940 | 1132503 |
| **884** | 781456 | 690807104 | 29.7321375 | 9.5973373 | 92.10888 | 1131222 |
| **885** | 783225 | 693154125 | 29.7489496 | 9.6009548 | 92.17833 | 1129944 |
| **886** | 784996 | 695506456 | 29.7657521 | 9.6045696 | 92.24776 | 1128668 |
| **887** | 786769 | 697864103 | 29.7825452 | 9.6081817 | 92.31715 | 1127396 |
| **888** | 788544 | 700227072 | 29.7993289 | 9.6117911 | 92.38653 | 1126126 |
| **889** | 790321 | 702595369 | 29.8161030 | 9.6153977 | 92.45587 | 1124859 |
| **890** | 792100 | 704969000 | 29.8328678 | 9.6190017 | 92.52519 | 1123596 |
| **891** | 793881 | 707347971 | 29.8496231 | 9.6226030 | 92.59449 | 1122334 |
| **892** | 795664 | 709732288 | 29.8663690 | 9.6262016 | 92.66376 | 1121076 |
| **893** | 797449 | 712121957 | 29.8831056 | 9.6297975 | 92.73300 | 1119821 |
| **894** | 799236 | 714516984 | 29.8998328 | 9.6333907 | 92.80222 | 1118568 |
| **895** | 801025 | 716917375 | 29.9165506 | 9.6369812 | 92.87141 | 1117318 |
| **896** | 802816 | 719323136 | 29.9332591 | 9.6405690 | 92.94057 | 1116071 |
| **897** | 804609 | 721734273 | 29.9499583 | 9.6441542 | 93.00971 | 1114827 |
| **898** | 806404 | 724150792 | 29.9666481 | 9.6477367 | 93.07882 | 1113586 |
| **899** | 808201 | 726572699 | 29.9833287 | 9.6513166 | 93.14791 | 1112347 |
| **900** | 810000 | 729000000 | 30.0000000 | 9.6548938 | 93.21698 | 1111111 |

## TABLE I.

| No. | Square | Cube | Square Root | Cube Root | Cube Root of the Square | Reciprocal 0.00 |
|---|---|---|---|---|---|---|
| **901** | 811801 | 731432701 | 30.0166620 | 9.6584684 | 93.28601 | 1109878 |
| **902** | 813604 | 733870808 | 30.0333148 | 9.6620403 | 93.35502 | 1108647 |
| **903** | 815409 | 736314327 | 30.0499584 | 9.6656096 | 93.42401 | 1107420 |
| **904** | 817216 | 738763264 | 30.0665928 | 9.6691762 | 93.49297 | 1106195 |
| **905** | 819025 | 741217625 | 30.0832179 | 9.6727403 | 93.56190 | 1104972 |
| **906** | 820836 | 743677416 | 30.0998339 | 9.6763017 | 93.63081 | 1103753 |
| **907** | 822649 | 746142643 | 30.1164407 | 9.6798604 | 93.69970 | 1102536 |
| **908** | 824464 | 748613312 | 30.1330383 | 9.6834166 | 93.76856 | 1101322 |
| **909** | 826281 | 751089429 | 30.1496269 | 9.6869701 | 93.83739 | 1100110 |
| **910** | 828100 | 753571000 | 30.1662063 | 9.6905211 | 93.90620 | 1098901 |
| **911** | 829921 | 756058031 | 30.1827765 | 9.6940694 | 93.97498 | 1097695 |
| **912** | 831744 | 758550528 | 30.1993377 | 9.6976151 | 94.04374 | 1096491 |
| **913** | 833569 | 761048497 | 30.2158899 | 9.7011583 | 94.11247 | 1095290 |
| **914** | 835396 | 763551944 | 30.2324329 | 9.7046989 | 94.18118 | 1094092 |
| **915** | 837225 | 766060875 | 30.2489669 | 9.7082369 | 94.24986 | 1092896 |
| **916** | 839056 | 768575296 | 30.2654919 | 9.7117723 | 94.31852 | 1091703 |
| **917** | 840889 | 771095213 | 30.2820079 | 9.7153051 | 94.38715 | 1090513 |
| **918** | 842724 | 773620632 | 30.2985148 | 9.7188354 | 94.45576 | 1089325 |
| **919** | 844561 | 776151559 | 30.3150128 | 9.7223631 | 94.52434 | 1088139 |
| **920** | 846400 | 778688000 | 30.3315018 | 9.7258883 | 94.59290 | 1086957 |
| **921** | 848241 | 781229961 | 30.3479818 | 9.7294109 | 94.66144 | 1085776 |
| **922** | 850084 | 783777448 | 30.3644529 | 9.7329309 | 94.72994 | 1084599 |
| **923** | 851929 | 786330467 | 30.3809151 | 9.7364484 | 94.79843 | 1083424 |
| **924** | 853776 | 788889024 | 30.3973683 | 9.7399634 | 94.86689 | 1082251 |
| **925** | 855625 | 791453125 | 30.4138127 | 9.7434758 | 94.93532 | 1081081 |
| **926** | 857476 | 794022776 | 30.4302481 | 9.7469857 | 95.00373 | 1079914 |
| **927** | 859329 | 796597983 | 30.4466747 | 9.7504930 | 95.07212 | 1078749 |
| **928** | 861184 | 799178752 | 30.4630924 | 9.7539979 | 95.14048 | 1077586 |
| **929** | 863041 | 801765089 | 30.4795013 | 9.7575002 | 95.20881 | 1076426 |
| **930** | 864900 | 804357000 | 30.4959014 | 9.7610001 | 95.27712 | 1075269 |
| **931** | 866761 | 806954491 | 30.5122926 | 9.7644974 | 95.34541 | 1074114 |
| **932** | 868624 | 809557568 | 30.5286750 | 9.7679922 | 95.41367 | 1072961 |
| **933** | 870489 | 812166237 | 30.5450487 | 9.7714845 | 95.48191 | 1071811 |
| **934** | 872356 | 814780504 | 30.5614136 | 9.7749743 | 95.55012 | 1070664 |
| **935** | 874225 | 817400375 | 30.5777697 | 9.7784616 | 95.61831 | 1069519 |
| **936** | 876096 | 820025856 | 30.5941171 | 9.7819466 | 95.68648 | 1068376 |
| **937** | 877969 | 822656953 | 30.6104557 | 9.7854288 | 95.75462 | 1067236 |
| **938** | 879844 | 825293672 | 30.6267857 | 9.7889087 | 95.82273 | 1066098 |
| **939** | 881721 | 827936019 | 30.6431069 | 9.7923861 | 95.89083 | 1064963 |
| **940** | 883600 | 830584000 | 30.6594194 | 9.7958611 | 95.95889 | 1063830 |
| **941** | 885481 | 833237621 | 30.6757233 | 9.7993336 | 96.02694 | 1062699 |
| **942** | 887364 | 835896888 | 30.6920185 | 9.8028036 | 96.09496 | 1061571 |
| **943** | 889249 | 838561807 | 30.7083051 | 9.8062711 | 96.16295 | 1060445 |
| **944** | 891136 | 841232384 | 30.7245830 | 9.8097362 | 96.23093 | 1059322 |
| **945** | 893025 | 843908625 | 30.7408523 | 9.8131989 | 96.29887 | 1058201 |
| **946** | 894916 | 846590536 | 30.7571130 | 9.8166591 | 96.36680 | 1057082 |
| **947** | 896809 | 849278123 | 30.7733651 | 9.8201169 | 96.43470 | 1055966 |
| **948** | 898704 | 851971392 | 30.7896086 | 9.8235723 | 96.50257 | 1054852 |
| **949** | 900601 | 854670349 | 30.8058436 | 9.8270252 | 96.57042 | 1053741 |
| **950** | 902500 | 857375000 | 30.8220700 | 9.8304757 | 96.63825 | 1052632 |

TABLE I.

| No. | Square | Cube | Square Root | Cube Root | Cube Root of the Square | Reciprocal 0.00 |
|---|---|---|---|---|---|---|
| **951** | 904401 | 860085351 | 30.8382879 | 9.8339238 | 96.70606 | 1051525 |
| **952** | 906304 | 862801408 | 30.8544972 | 9.8373695 | 96.77384 | 1050420 |
| **953** | 908209 | 865523177 | 30.8706981 | 9.8408127 | 96.84159 | 1049318 |
| **954** | 910116 | 868250664 | 30.8868904 | 9.8442536 | 96.90933 | 1048218 |
| **955** | 912025 | 870983875 | 30.9030743 | 9.8476920 | 96.97704 | 1047120 |
| **956** | 913936 | 873722816 | 30.9192497 | 9.8511280 | 97.04472 | 1046025 |
| **957** | 915849 | 876467493 | 30.9354166 | 9.8545617 | 97.11239 | 1044932 |
| **958** | 917764 | 879217912 | 30.9515751 | 9.8579929 | 97.18002 | 1043841 |
| **959** | 919681 | 881974079 | 30.9677251 | 9.8614218 | 97.24764 | 1042753 |
| **960** | 921600 | 884736000 | 30.9838668 | 9.8648483 | 97.31523 | 1041667 |
| **961** | 923521 | 887503681 | 31.0000000 | 9.8682724 | 97.38280 | 1040583 |
| **962** | 925444 | 890277128 | 31.0161248 | 9.8716941 | 97.45035 | 1039501 |
| **963** | 927369 | 893056347 | 31.0322413 | 9.8751135 | 97.51787 | 1038422 |
| **964** | 929296 | 895841344 | 31.0483494 | 9.8785305 | 97.58536 | 1037344 |
| **965** | 931225 | 898632125 | 31.0644491 | 9.8819451 | 97.65284 | 1036269 |
| **966** | 933156 | 901428696 | 31.0805405 | 9.8853574 | 97.72029 | 1035197 |
| **967** | 935089 | 904231063 | 31.0966236 | 9.8887673 | 97.78772 | 1034126 |
| **968** | 937024 | 907039232 | 31.1126984 | 9.8921749 | 97.85512 | 1033058 |
| **969** | 938961 | 909853209 | 31.1287648 | 9.8955801 | 97.92251 | 1031992 |
| **970** | 940900 | 912673000 | 31.1448230 | 9.8989830 | 97.98987 | 1030928 |
| **971** | 942841 | 915498611 | 31.1608729 | 9.9023835 | 98.05720 | 1029866 |
| **972** | 944784 | 918330048 | 31.1769145 | 9.9057817 | 98.12451 | 1028807 |
| **973** | 946729 | 921167317 | 31.1929479 | 9.9091776 | 98.19180 | 1027749 |
| **974** | 948676 | 924010424 | 31.2089731 | 9.9125712 | 98.25907 | 1026694 |
| **975** | 950625 | 926859375 | 31.2249900 | 9.9159624 | 98.32631 | 1025641 |
| **976** | 952576 | 929714176 | 31.2409987 | 9.9193513 | 98.39354 | 1024590 |
| **977** | 954529 | 932574833 | 31.2569992 | 9.9227379 | 98.46073 | 1023541 |
| **978** | 956484 | 935441352 | 31.2729915 | 9.9261222 | 98.52790 | 1022495 |
| **979** | 958441 | 938313739 | 31.2889757 | 9.9295042 | 98.59505 | 1021450 |
| **980** | 960400 | 941192000 | 31.3049517 | 9.9328839 | 98.66218 | 1020408 |
| **981** | 962361 | 944076141 | 31.3209195 | 9.9362613 | 98.72929 | 1019368 |
| **982** | 964324 | 946966168 | 31.3368792 | 9.9396363 | 98.79637 | 1018330 |
| **983** | 966289 | 949862087 | 31.3528308 | 9.9430092 | 98.86343 | 1017294 |
| **984** | 968256 | 952763904 | 31.3687743 | 9.9463797 | 98.93047 | 1016260 |
| **985** | 970225 | 955671625 | 31.3847097 | 9.9497479 | 98.99748 | 1015228 |
| **986** | 972196 | 958585256 | 31.4006369 | 9.9531138 | 99.06448 | 1014199 |
| **987** | 974169 | 961504803 | 31.4165561 | 9.9564775 | 99.13144 | 1013171 |
| **988** | 976144 | 964430272 | 31.4324673 | 9.9598389 | 99.19839 | 1012146 |
| **989** | 978121 | 967361669 | 31.4483704 | 9.9631981 | 99.26532 | 1011122 |
| **990** | 980100 | 970299000 | 31.4642654 | 9.9665549 | 99.33222 | 1010101 |
| **991** | 982081 | 973242271 | 31.4801525 | 9.9699095 | 99.39910 | 1009082 |
| **992** | 984064 | 976191488 | 31.4960315 | 9.9732619 | 99.46595 | 1008065 |
| **993** | 986049 | 979146657 | 31.5119025 | 9.9766120 | 99.53279 | 1007049 |
| **994** | 988036 | 982107784 | 31.5277655 | 9.9799599 | 99.59960 | 1006036 |
| **995** | 990025 | 985074875 | 31.5436206 | 9.9833055 | 99.66639 | 1005025 |
| **996** | 992016 | 988047936 | 31.5594677 | 9.9866488 | 99.73316 | 1004016 |
| **997** | 994009 | 991026973 | 31.5753068 | 9.9899900 | 99.79990 | 1003009 |
| **998** | 996004 | 994011992 | 31.5911380 | 9.9933289 | 99.86662 | 1002004 |
| **999** | 998001 | 997002999 | 31.6069613 | 9.9966656 | 99.93332 | 1001001 |
| **1000** | 1000000 | 1000000000 | 31.6227766 | 10.0000000 | 100.00000 | 1000000 |

# COMMON LOGARITHMS

OF

# NUMBERS

FROM 100 TO 999

## TABLE II.—LOGARITHMS OF NUMBERS.

| No. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 100 | 00 000 | 043 | 087 | 130 | 173 | 217 | 260 | 303 | 346 | 389 |
| 101 | 432 | 475 | 518 | 561 | 604 | 647 | 689 | 732 | 775 | 817 |
| 102 | 860 | 903 | 945 | 988 | *030 | *072 | *115 | *157 | *199 | *242 |
| 103 | 01 284 | 326 | 368 | 410 | 452 | 494 | 536 | 578 | 620 | 662 |
| 104 | 703 | 745 | 787 | 828 | 870 | 912 | 953 | 995 | *036 | *078 |
| 105 | 02 119 | 160 | 202 | 243 | 284 | 325 | 366 | 407 | 449 | 490 |
| 106 | 531 | 572 | 612 | 653 | 694 | 735 | 776 | 816 | 857 | 898 |
| 107 | 938 | 979 | *019 | *060 | *100 | *141 | *181 | *222 | *262 | *302 |
| 108 | 03 342 | 383 | 423 | 463 | 503 | 543 | 583 | 623 | 663 | 703 |
| 109 | 743 | 782 | 822 | 862 | 902 | 941 | 981 | *021 | *060 | *100 |
| 110 | 04 139 | 179 | 218 | 258 | 297 | 336 | 376 | 415 | 454 | 493 |
| 111 | 532 | 571 | 610 | 650 | 689 | 727 | 766 | 805 | 844 | 883 |
| 112 | 922 | 961 | 999 | *038 | *077 | *115 | *154 | *192 | *231 | *269 |
| 113 | 05 308 | 346 | 385 | 423 | 461 | 500 | 538 | 576 | 614 | 652 |
| 114 | 690 | 729 | 767 | 805 | 843 | 881 | 918 | 956 | 994 | *032 |
| 115 | 06 070 | 108 | 145 | 183 | 221 | 258 | 296 | 333 | 371 | 408 |
| 116 | 446 | 483 | 521 | 558 | 595 | 633 | 670 | 707 | 744 | 781 |
| 117 | 819 | 856 | 893 | 930 | 967 | *004 | *041 | *078 | *115 | *151 |
| 118 | 07 188 | 225 | 262 | 298 | 335 | 372 | 408 | 445 | 482 | 518 |
| 119 | 555 | 591 | 628 | 664 | 700 | 737 | 773 | 809 | 846 | 882 |
| 120 | 918 | 954 | 990 | *027 | *063 | *099 | *135 | *171 | *207 | *243 |
| 121 | 08 279 | 314 | 350 | 386 | 422 | 458 | 493 | 529 | 565 | 600 |
| 122 | 636 | 672 | 707 | 743 | 778 | 814 | 849 | 884 | 920 | 955 |
| 123 | 991 | *026 | *061 | *096 | *132 | *167 | *202 | *237 | *272 | *307 |
| 124 | 09 342 | 377 | 412 | 447 | 482 | 517 | 552 | 587 | 621 | 656 |
| 125 | 691 | 726 | 760 | 795 | 830 | 864 | 899 | 934 | 968 | *003 |
| 126 | 10 037 | 072 | 106 | 140 | 175 | 209 | 243 | 278 | 312 | 346 |
| 127 | 380 | 415 | 449 | 483 | 517 | 551 | 585 | 619 | 653 | 687 |
| 128 | 721 | 755 | 789 | 823 | 857 | 890 | 924 | 958 | 992 | *025 |
| 129 | 11 059 | 093 | 126 | 160 | 193 | 227 | 261 | 294 | 327 | 361 |
| 130 | 394 | 428 | 461 | 494 | 528 | 561 | 594 | 628 | 661 | 694 |
| 131 | 727 | 760 | 793 | 826 | 860 | 893 | 926 | 959 | 992 | *024 |
| 132 | 12 057 | 090 | 123 | 156 | 189 | 222 | 254 | 287 | 320 | 352 |
| 133 | 385 | 418 | 450 | 483 | 516 | 548 | 581 | 613 | 646 | 678 |
| 134 | 710 | 743 | 775 | 808 | 840 | 872 | 905 | 937 | 969 | *001 |
| 135 | 13 033 | 066 | 098 | 130 | 162 | 194 | 226 | 258 | 290 | 322 |
| 136 | 354 | 386 | 418 | 450 | 481 | 513 | 545 | 577 | 609 | 640 |
| 137 | 672 | 704 | 735 | 767 | 799 | 830 | 862 | 893 | 925 | 956 |
| 138 | 988 | *019 | *051 | *082 | *114 | *145 | *176 | *208 | *239 | *270 |
| 139 | 14 301 | 333 | 364 | 395 | 426 | 457 | 489 | 520 | 551 | 582 |
| 140 | 613 | 644 | 675 | 706 | 737 | 768 | 799 | 829 | 860 | 891 |
| 141 | 922 | 953 | 983 | *014 | *045 | *076 | *106 | *137 | *168 | *198 |
| 142 | 15 229 | 259 | 290 | 320 | 351 | 381 | 412 | 442 | 473 | 503 |
| 143 | 534 | 564 | 594 | 625 | 655 | 685 | 715 | 746 | 776 | 806 |
| 144 | 836 | 866 | 897 | 927 | 957 | 987 | *017 | *047 | *077 | *107 |
| 145 | 16 137 | 167 | 197 | 227 | 256 | 286 | 316 | 346 | 376 | 406 |
| 146 | 435 | 465 | 495 | 524 | 554 | 584 | 613 | 643 | 673 | 702 |
| 147 | 732 | 761 | 791 | 820 | 850 | 879 | 909 | 938 | 967 | 997 |
| 148 | 17 026 | 056 | 085 | 114 | 143 | 173 | 202 | 231 | 260 | 289 |
| 149 | 319 | 348 | 377 | 406 | 435 | 464 | 403 | 522 | 551 | 580 |

**Pp. Pts.**

| | 44 | 43 | 42 |
|---|---|---|---|
| 1 | 4.4 | 4.3 | 4.2 |
| 2 | 8.8 | 8.6 | 8.4 |
| 3 | 13.2 | 12.9 | 12.6 |
| 4 | 17.6 | 17.2 | 16.8 |
| 5 | 22.0 | 21.5 | 21.0 |
| 6 | 26.4 | 25.8 | 25.2 |
| 7 | 30.8 | 30.1 | 29.4 |
| 8 | 35.2 | 34.4 | 33.6 |
| 9 | 39.6 | 38.7 | 37.8 |

| | 41 | 40 | 39 |
|---|---|---|---|
| 1 | 4.1 | 4.0 | 3.9 |
| 2 | 8.2 | 8.0 | 7.8 |
| 3 | 12.3 | 12.0 | 11.7 |
| 4 | 16.4 | 16.0 | 15.6 |
| 5 | 20.5 | 20.0 | 19.5 |
| 6 | 24.6 | 24.0 | 23.4 |
| 7 | 28.7 | 28.0 | 27.3 |
| 8 | 32.8 | 32.0 | 31.2 |
| 9 | 36.9 | 36.0 | 35.1 |

| | 38 | 37 | 36 |
|---|---|---|---|
| 1 | 3.8 | 3.7 | 3.6 |
| 2 | 7.6 | 7.4 | 7.2 |
| 3 | 11.4 | 11.1 | 10.8 |
| 4 | 15.2 | 14.8 | 14.4 |
| 5 | 19.0 | 18.5 | 18.0 |
| 6 | 22.8 | 22.2 | 21.6 |
| 7 | 26.6 | 25.9 | 25.2 |
| 8 | 30.4 | 29.6 | 28.8 |
| 9 | 34.2 | 33.3 | 32.4 |

| | 35 | 34 | 33 |
|---|---|---|---|
| 1 | 3.5 | 3.4 | 3.3 |
| 2 | 7.0 | 6.8 | 6.6 |
| 3 | 10.5 | 10.2 | 9.9 |
| 4 | 14.0 | 13.6 | 13.2 |
| 5 | 17.5 | 17.0 | 16.5 |
| 6 | 21.0 | 20.4 | 19.8 |
| 7 | 24.5 | 23.8 | 23.1 |
| 8 | 28.0 | 27.2 | 26.4 |
| 9 | 31.5 | 30.6 | 29.7 |

| | 32 | 31 | 30 |
|---|---|---|---|
| 1 | 3.2 | 3.1 | 3.0 |
| 2 | 6.4 | 6.2 | 6.0 |
| 3 | 9.6 | 9.3 | 9.0 |
| 4 | 12.8 | 12.4 | 12.0 |
| 5 | 16.0 | 15.5 | 15.0 |
| 6 | 19.2 | 18.6 | 18.0 |
| 7 | 22.4 | 21.7 | 21.0 |
| 8 | 25.6 | 24.8 | 24.0 |
| 9 | 28.8 | 27.9 | 27.0 |

## TABLE II.—LOGARITHMS OF NUMBERS.

| No. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 150 | 17 609 | 638 | 667 | 696 | 725 | 754 | 782 | 811 | 840 | 869 |
| 151 | 898 | 926 | 955 | 984 | *013 | *041 | *070 | *099 | *127 | *156 |
| 152 | 18 184 | 213 | 241 | 270 | 298 | 327 | 355 | 384 | 412 | 441 |
| 153 | 469 | 498 | 526 | 554 | 583 | 611 | 639 | 667 | 696 | 724 |
| 154 | 752 | 780 | 808 | 837 | 865 | 893 | 921 | 949 | 977 | *005 |
| 155 | 19 033 | 061 | 089 | 117 | 145 | 173 | 201 | 229 | 257 | 285 |
| 156 | 312 | 340 | 368 | 396 | 424 | 451 | 479 | 507 | 535 | 562 |
| 157 | 590 | 618 | 645 | 673 | 700 | 728 | 756 | 783 | 811 | 838 |
| 158 | 866 | 893 | 921 | 948 | 976 | *003 | *030 | *058 | *085 | *112 |
| 159 | 20 140 | 167 | 194 | 222 | 249 | 276 | 303 | 330 | 358 | 385 |
| 160 | 412 | 439 | 466 | 493 | 520 | 548 | 575 | 602 | 629 | 656 |
| 161 | 683 | 710 | 737 | 763 | 790 | 817 | 844 | 871 | 898 | 925 |
| 162 | 952 | 978 | *005 | *032 | *059 | *085 | *112 | *139 | *165 | *192 |
| 163 | 21 219 | 245 | 272 | 299 | 325 | 352 | 378 | 405 | 431 | 458 |
| 164 | 484 | 511 | 537 | 564 | 590 | 617 | 643 | 669 | 696 | 722 |
| 165 | 748 | 775 | 801 | 827 | 854 | 880 | 906 | 932 | 958 | 985 |
| 166 | 22 011 | 037 | 063 | 089 | 115 | 141 | 167 | 194 | 220 | 246 |
| 167 | 272 | 298 | 324 | 350 | 376 | 401 | 427 | 453 | 479 | 505 |
| 168 | 531 | 557 | 583 | 608 | 634 | 660 | 686 | 712 | 737 | 763 |
| 169 | 789 | 814 | 840 | 866 | 891 | 917 | 943 | 968 | 994 | *019 |
| 170 | 23 045 | 070 | 096 | 121 | 147 | 172 | 198 | 223 | 249 | 274 |
| 171 | 300 | 325 | 350 | 376 | 401 | 426 | 452 | 477 | 502 | 528 |
| 172 | 553 | 578 | 603 | 629 | 654 | 679 | 704 | 729 | 754 | 779 |
| 173 | 805 | 830 | 855 | 880 | 905 | 930 | 955 | 980 | *005 | *030 |
| 174 | 24 055 | 080 | 105 | 130 | 155 | 180 | 204 | 229 | 254 | 279 |
| 175 | 304 | 329 | 353 | 378 | 403 | 428 | 452 | 477 | 502 | 527 |
| 176 | 551 | 576 | 601 | 625 | 650 | 674 | 699 | 724 | 748 | 773 |
| 177 | 797 | 822 | 846 | 871 | 895 | 920 | 944 | 969 | 993 | *018 |
| 178 | 25 042 | 066 | 091 | 115 | 139 | 164 | 188 | 212 | 237 | 261 |
| 179 | 285 | 310 | 334 | 358 | 382 | 406 | 431 | 455 | 479 | 503 |
| 180 | 527 | 551 | 575 | 600 | 624 | 648 | 672 | 696 | 720 | 744 |
| 181 | 768 | 792 | 816 | 840 | 864 | 888 | 912 | 935 | 959 | 983 |
| 182 | 26 007 | 031 | 055 | 079 | 102 | 126 | 150 | 174 | 198 | 221 |
| 183 | 245 | 269 | 293 | 316 | 340 | 364 | 387 | 411 | 435 | 458 |
| 184 | 482 | 505 | 529 | 553 | 576 | 600 | 623 | 647 | 670 | 694 |
| 185 | 717 | 741 | 764 | 788 | 811 | 834 | 858 | 881 | 905 | 928 |
| 186 | 951 | 975 | 998 | *021 | *045 | *068 | *091 | *114 | *138 | *161 |
| 187 | 27 184 | 207 | 231 | 254 | 277 | 300 | 323 | 346 | 370 | 393 |
| 188 | 416 | 439 | 462 | 485 | 508 | 531 | 554 | 577 | 600 | 623 |
| 189 | 646 | 669 | 692 | 715 | 738 | 761 | 784 | 807 | 830 | 852 |
| 190 | 875 | 898 | 921 | 944 | 967 | 989 | *012 | *035 | *058 | *081 |
| 191 | 28 103 | 126 | 149 | 171 | 194 | 217 | 240 | 262 | 285 | 307 |
| 192 | 330 | 353 | 375 | 398 | 421 | 443 | 466 | 488 | 511 | 533 |
| 193 | 556 | 578 | 601 | 623 | 646 | 668 | 691 | 713 | 735 | 758 |
| 194 | 780 | 803 | 825 | 847 | 870 | 892 | 914 | 937 | 959 | 981 |
| 195 | 29 003 | 026 | 048 | 070 | 092 | 115 | 137 | 159 | 181 | 203 |
| 196 | 226 | 248 | 270 | 292 | 314 | 336 | 358 | 380 | 403 | 425 |
| 197 | 447 | 469 | 491 | 513 | 535 | 557 | 579 | 601 | 623 | 645 |
| 198 | 667 | 688 | 710 | 732 | 754 | 776 | 798 | 820 | 842 | 863 |
| 199 | 885 | 907 | 929 | 951 | 973 | 994 | *016 | *038 | *060 | *081 |
| | 30 | | | | | | | | | |

**Pp. Pts.**

| | 29 | 28 |
|---|---|---|
| 1 | 2.9 | 2.8 |
| 2 | 5.8 | 5.6 |
| 3 | 8.7 | 8.4 |
| 4 | 11.6 | 11.2 |
| 5 | 14.5 | 14.0 |
| 6 | 17.4 | 16.8 |
| 7 | 20.3 | 19.6 |
| 8 | 23.2 | 22.4 |
| 9 | 26.1 | 25.2 |

| | 27 | 26 |
|---|---|---|
| 1 | 2.7 | 2.6 |
| 2 | 5.4 | 5.2 |
| 3 | 8.1 | 7.8 |
| 4 | 10.8 | 10.4 |
| 5 | 13.5 | 13.0 |
| 6 | 16.2 | 15.6 |
| 7 | 18.9 | 18.2 |
| 8 | 21.6 | 20.8 |
| 9 | 24.3 | 23.4 |

| | 25 |
|---|---|
| 1 | 2.5 |
| 2 | 5.0 |
| 3 | 7.5 |
| 4 | 10.0 |
| 5 | 12.5 |
| 6 | 15.0 |
| 7 | 17.5 |
| 8 | 20.0 |
| 9 | 22.5 |

| | 24 | 23 |
|---|---|---|
| 1 | 2.4 | 2.3 |
| 2 | 4.8 | 4.6 |
| 3 | 7.2 | 6.9 |
| 4 | 9.6 | 9.2 |
| 5 | 12.0 | 11.5 |
| 6 | 14.4 | 13.8 |
| 7 | 16.8 | 16.1 |
| 8 | 19.2 | 18.4 |
| 9 | 21.6 | 20.7 |

| | 22 | 21 |
|---|---|---|
| 1 | 2.2 | 2.1 |
| 2 | 4.4 | 4.2 |
| 3 | 6.6 | 6.3 |
| 4 | 8.8 | 8.4 |
| 5 | 11.0 | 10.5 |
| 6 | 13.2 | 12.6 |
| 7 | 15.4 | 14.7 |
| 8 | 17.6 | 16.8 |
| 9 | 19.8 | 18.9 |

| No. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 200 | 30 103 | 125 | 146 | 168 | 190 | 211 | 233 | 255 | 276 | 298 |
| 201 | 320 | 341 | 363 | 384 | 406 | 428 | 449 | 471 | 492 | 514 |
| 202 | 535 | 557 | 578 | 600 | 621 | 643 | 664 | 685 | 707 | 728 |
| 203 | 750 | 771 | 792 | 814 | 835 | 856 | 878 | 899 | 920 | 942 |
| 204 | 963 | 984 | *006 | *027 | *048 | *069 | *091 | *112 | *133 | *154 |
| 205 | 31 175 | 197 | 218 | 239 | 260 | 281 | 302 | 323 | 345 | 366 |
| 206 | 387 | 408 | 429 | 450 | 471 | 492 | 513 | 534 | 555 | 576 |
| 207 | 597 | 618 | 639 | 660 | 681 | 702 | 723 | 744 | 765 | 785 |
| 208 | 806 | 827 | 848 | 869 | 890 | 911 | 931 | 952 | 973 | 994 |
| 209 | 32 015 | 035 | 056 | 077 | 098 | 118 | 139 | 160 | 181 | 201 |
| 210 | 222 | 243 | 263 | 284 | 305 | 325 | 346 | 366 | 387 | 408 |
| 211 | 428 | 449 | 469 | 490 | 510 | 531 | 552 | 572 | 593 | 613 |
| 212 | 634 | 654 | 675 | 695 | 715 | 736 | 756 | 777 | 797 | 818 |
| 213 | 838 | 858 | 879 | 899 | 919 | 940 | 960 | 980 | *001 | *021 |
| 214 | 33 041 | 062 | 082 | 102 | 122 | 143 | 163 | 183 | 203 | 224 |
| 215 | 244 | 264 | 284 | 304 | 325 | 345 | 365 | 385 | 405 | 425 |
| 216 | 445 | 465 | 486 | 506 | 526 | 546 | 566 | 586 | 606 | 626 |
| 217 | 646 | 666 | 686 | 706 | 726 | 746 | 766 | 786 | 806 | 826 |
| 218 | 846 | 866 | 885 | 905 | 925 | 945 | 965 | 985 | *005 | *025 |
| 219 | 34 044 | 064 | 084 | 104 | 124 | 143 | 163 | 183 | 203 | 223 |
| 220 | 242 | 262 | 282 | 301 | 321 | 341 | 361 | 380 | 400 | 420 |
| 221 | 439 | 459 | 479 | 498 | 518 | 537 | 557 | 577 | 596 | 616 |
| 222 | 635 | 655 | 674 | 694 | 713 | 733 | 753 | 772 | 792 | 811 |
| 223 | 830 | 850 | 869 | 889 | 908 | 928 | 947 | 967 | 986 | *005 |
| 224 | 35 025 | 044 | 064 | 083 | 102 | 122 | 141 | 160 | 180 | 199 |
| 225 | 218 | 238 | 257 | 276 | 295 | 315 | 334 | 353 | 372 | 392 |
| 226 | 411 | 430 | 449 | 468 | 488 | 507 | 526 | 545 | 564 | 583 |
| 227 | 603 | 622 | 641 | 660 | 679 | 698 | 717 | 736 | 755 | 774 |
| 228 | 793 | 813 | 832 | 851 | 870 | 889 | 908 | 927 | 946 | 965 |
| 229 | 984 | *003 | *021 | *040 | *059 | *078 | *097 | *116 | *135 | *154 |
| 230 | 36 173 | 192 | 211 | 229 | 248 | 267 | 286 | 305 | 324 | 342 |
| 231 | 361 | 380 | 399 | 418 | 436 | 455 | 474 | 493 | 511 | 530 |
| 232 | 549 | 568 | 586 | 605 | 624 | 642 | 661 | 680 | 698 | 717 |
| 233 | 736 | 754 | 773 | 791 | 810 | 829 | 847 | 866 | 884 | 903 |
| 234 | 922 | 940 | 959 | 977 | 996 | *014 | *033 | *051 | *070 | *088 |
| 235 | 37 107 | 125 | 144 | 162 | 181 | 199 | 218 | 236 | 254 | 273 |
| 236 | 291 | 310 | 328 | 346 | 365 | 383 | 401 | 420 | 438 | 457 |
| 237 | 475 | 493 | 511 | 530 | 548 | 566 | 585 | 603 | 621 | 639 |
| 238 | 658 | 676 | 694 | 712 | 731 | 749 | 767 | 785 | 803 | 822 |
| 239 | 840 | 858 | 876 | 894 | 912 | 931 | 949 | 967 | 985 | *003 |
| 240 | 38 021 | 039 | 057 | 075 | 093 | 112 | 130 | 148 | 166 | 184 |
| 241 | 202 | 220 | 238 | 256 | 274 | 292 | 310 | 328 | 346 | 364 |
| 242 | 382 | 399 | 417 | 435 | 453 | 471 | 489 | 507 | 525 | 543 |
| 243 | 561 | 578 | 596 | 614 | 632 | 650 | 668 | 686 | 703 | 721 |
| 244 | 739 | 757 | 775 | 792 | 810 | 828 | 846 | 863 | 881 | 899 |
| 245 | 917 | 934 | 952 | 970 | 987 | *005 | *023 | *041 | *058 | *076 |
| 246 | 39 094 | 111 | 129 | 146 | 164 | 182 | 199 | 217 | 235 | 252 |
| 247 | 270 | 287 | 305 | 322 | 340 | 358 | 375 | 393 | 410 | 428 |
| 248 | 445 | 463 | 480 | 498 | 515 | 533 | 550 | 568 | 585 | 602 |
| 249 | 620 | 637 | 655 | 672 | 690 | 707 | 724 | 742 | 759 | 777 |

Pp. Pts.

| | 22 | 21 | 20 | 19 | 18 | 17 |
|---|---|---|---|---|---|---|
| 1 | 2.2 | 2.1 | 2.0 | 1.9 | 1.8 | 1.7 |
| 2 | 4.4 | 4.2 | 4.0 | 3.8 | 3.6 | 3.4 |
| 3 | 6.6 | 6.3 | 6.0 | 5.7 | 5.4 | 5.1 |
| 4 | 8.8 | 8.4 | 8.0 | 7.6 | 7.2 | 6.8 |
| 5 | 11.0 | 10.5 | 10.0 | 9.5 | 9.0 | 8.5 |
| 6 | 13.2 | 12.6 | 12.0 | 11.4 | 10.8 | 10.2 |
| 7 | 15.4 | 14.7 | 14.0 | 13.3 | 12.6 | 11.9 |
| 8 | 17.6 | 16.8 | 16.0 | 15.2 | 14.4 | 13.6 |
| 9 | 19.8 | 18.9 | 18.0 | 17.1 | 16.2 | 15.3 |

## TABLE II.—LOGARITHMS OF NUMBERS.

| No. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 250 | 39 794 | 811 | 829 | 846 | 863 | 881 | 898 | 915 | 933 | 950 |
| 251 | 967 | 985 | *002 | *019 | *037 | *054 | *071 | *088 | *106 | *123 |
| 252 | 40 140 | 157 | 175 | 192 | 209 | 226 | 243 | 261 | 278 | 295 |
| 253 | 312 | 329 | 346 | 364 | 381 | 398 | 415 | 432 | 449 | 466 |
| 254 | 483 | 500 | 518 | 535 | 552 | 569 | 586 | 603 | 620 | 637 |
| 255 | 654 | 671 | 688 | 705 | 722 | 739 | 756 | 773 | 790 | 807 |
| 256 | 824 | 841 | 858 | 875 | 892 | 909 | 926 | 943 | 960 | 976 |
| 257 | 993 | *010 | *027 | *044 | *061 | *078 | *095 | *111 | *128 | *145 |
| 258 | 41 162 | 179 | 196 | 212 | 229 | 246 | 263 | 280 | 296 | 313 |
| 259 | 330 | 347 | 363 | 380 | 397 | 414 | 430 | 447 | 464 | 481 |
| 260 | 497 | 514 | 531 | 547 | 564 | 581 | 597 | 614 | 631 | 647 |
| 261 | 664 | 681 | 697 | 714 | 731 | 747 | 764 | 780 | 797 | 814 |
| 262 | 830 | 847 | 863 | 880 | 896 | 913 | 929 | 946 | 963 | 979 |
| 263 | 996 | *012 | *029 | *045 | *062 | *078 | *095 | *111 | *127 | *144 |
| 264 | 42 160 | 177 | 193 | 210 | 226 | 243 | 259 | 275 | 292 | 308 |
| 265 | 325 | 341 | 357 | 374 | 390 | 406 | 423 | 439 | 455 | 472 |
| 266 | 488 | 504 | 521 | 537 | 553 | 570 | 586 | 602 | 619 | 635 |
| 267 | 651 | 667 | 684 | 700 | 716 | 732 | 749 | 765 | 781 | 797 |
| 268 | 813 | 830 | 846 | 862 | 878 | 894 | 911 | 927 | 943 | 959 |
| 269 | 975 | 991 | *008 | *024 | *040 | *056 | *072 | *088 | *104 | *120 |
| 270 | 43 136 | 152 | 169 | 185 | 201 | 217 | 233 | 249 | 265 | 281 |
| 271 | 297 | 313 | 329 | 345 | 361 | 377 | 393 | 409 | 425 | 441 |
| 272 | 457 | 473 | 489 | 505 | 521 | 537 | 553 | 569 | 584 | 600 |
| 273 | 616 | 632 | 648 | 664 | 680 | 696 | 712 | 727 | 743 | 759 |
| 274 | 775 | 791 | 807 | 823 | 838 | 854 | 870 | 886 | 902 | 917 |
| 275 | 933 | 949 | 965 | 981 | 996 | *012 | *028 | *044 | *059 | *075 |
| 276 | 44 091 | 107 | 122 | 138 | 154 | 170 | 185 | 201 | 217 | 232 |
| 277 | 248 | 264 | 279 | 295 | 311 | 326 | 342 | 358 | 373 | 389 |
| 278 | 404 | 420 | 436 | 451 | 467 | 483 | 498 | 514 | 529 | 545 |
| 279 | 560 | 576 | 592 | 607 | 623 | 638 | 654 | 669 | 685 | 700 |
| 280 | 716 | 731 | 747 | 762 | 778 | 793 | 809 | 824 | 840 | 855 |
| 281 | 871 | 886 | 902 | 917 | 932 | 948 | 963 | 979 | 994 | *010 |
| 282 | 45 025 | 040 | 056 | 071 | 086 | 102 | 117 | 133 | 148 | 163 |
| 283 | 179 | 194 | 209 | 225 | 240 | 255 | 271 | 286 | 301 | 317 |
| 284 | 332 | 347 | 362 | 378 | 393 | 408 | 423 | 439 | 454 | 469 |
| 285 | 484 | 500 | 515 | 530 | 545 | 561 | 576 | 591 | 606 | 621 |
| 286 | 637 | 652 | 667 | 682 | 697 | 712 | 728 | 743 | 758 | 773 |
| 287 | 788 | 803 | 818 | 834 | 849 | 864 | 879 | 894 | 909 | 924 |
| 288 | 939 | 954 | 969 | 984 | *000 | *015 | *030 | *045 | *060 | *075 |
| 289 | 46 090 | 105 | 120 | 135 | 150 | 165 | 180 | 195 | 210 | 225 |
| 290 | 240 | 255 | 270 | 285 | 300 | 315 | 330 | 345 | 359 | 374 |
| 291 | 389 | 404 | 419 | 434 | 449 | 464 | 479 | 494 | 509 | 523 |
| 292 | 538 | 553 | 568 | 583 | 598 | 613 | 627 | 642 | 657 | 672 |
| 293 | 687 | 702 | 716 | 731 | 746 | 761 | 776 | 790 | 805 | 820 |
| 294 | 835 | 850 | 864 | 879 | 894 | 909 | 923 | 938 | 953 | 967 |
| 295 | 982 | 997 | *012 | *026 | *041 | *056 | *070 | *085 | *100 | *114 |
| 296 | 47 129 | 144 | 159 | 173 | 188 | 202 | 217 | 232 | 246 | 261 |
| 297 | 276 | 290 | 305 | 319 | 334 | 349 | 363 | 378 | 392 | 407 |
| 298 | 422 | 436 | 451 | 465 | 480 | 494 | 509 | 524 | 538 | 553 |
| 299 | 567 | 582 | 596 | 611 | 625 | 640 | 654 | 669 | 683 | 698 |

**Pp. Pts.**

| | 18 | 17 | 16 | 15 | 14 |
|---|---|---|---|---|---|
| 1 | 1.8 | 1.7 | 1.6 | 1.5 | 1.4 |
| 2 | 3.6 | 3.4 | 3.2 | 3.0 | 2.8 |
| 3 | 5.4 | 5.1 | 4.8 | 4.5 | 4.2 |
| 4 | 7.2 | 6.8 | 6.4 | 6.0 | 5.6 |
| 5 | 9.0 | 8.5 | 8.0 | 7.5 | 7.0 |
| 6 | 10.8 | 10.2 | 9.6 | 9.0 | 8.4 |
| 7 | 12.6 | 11.9 | 11.2 | 10.5 | 9.8 |
| 8 | 14.4 | 13.6 | 12.8 | 12.0 | 11.2 |
| 9 | 16.2 | 15.3 | 14.4 | 13.5 | 12.6 |

# TABLE II.—LOGARITHMS OF NUMBERS.

| No. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 300 | 47 712 | 727 | 741 | 756 | 770 | 784 | 799 | 813 | 828 | 842 |
| 301 | 857 | 871 | 885 | 900 | 914 | 929 | 943 | 958 | 972 | 986 |
| 302 | 48 001 | 015 | 029 | 044 | 058 | 073 | 087 | 101 | 116 | 130 |
| 303 | 144 | 159 | 173 | 187 | 202 | 216 | 230 | 244 | 259 | 273 |
| 304 | 287 | 302 | 316 | 330 | 344 | 359 | 373 | 387 | 401 | 416 |
| 305 | 430 | 444 | 458 | 473 | 487 | 501 | 515 | 530 | 544 | 558 |
| 306 | 572 | 586 | 601 | 615 | 629 | 643 | 657 | 671 | 686 | 700 |
| 307 | 714 | 728 | 742 | 756 | 770 | 785 | 799 | 813 | 827 | 841 |
| 308 | 855 | 869 | 883 | 897 | 911 | 926 | 940 | 954 | 968 | 982 |
| 309 | 996 | *010 | *024 | *038 | *052 | *066 | *080 | *094 | *108 | *122 |
| 310 | 49 136 | 150 | 164 | 178 | 192 | 206 | 220 | 234 | 248 | 262 |
| 311 | 276 | 290 | 304 | 318 | 332 | 346 | 360 | 374 | 388 | 402 |
| 312 | 415 | 429 | 443 | 457 | 471 | 485 | 499 | 513 | 527 | 541 |
| 313 | 554 | 568 | 582 | 596 | 610 | 624 | 638 | 651 | 665 | 679 |
| 314 | 693 | 707 | 721 | 734 | 748 | 762 | 776 | 790 | 803 | 817 |
| 315 | 831 | 845 | 859 | 872 | 886 | 900 | 914 | 927 | 941 | 955 |
| 316 | 969 | 982 | 996 | *010 | *024 | *037 | *051 | *065 | *079 | *092 |
| 317 | 50 106 | 120 | 133 | 147 | 161 | 174 | 188 | 202 | 215 | 229 |
| 318 | 243 | 256 | 270 | 284 | 297 | 311 | 325 | 338 | 352 | 365 |
| 319 | 379 | 393 | 406 | 420 | 433 | 447 | 461 | 474 | 488 | 501 |
| 320 | 515 | 529 | 542 | 556 | 569 | 583 | 596 | 610 | 623 | 637 |
| 321 | 651 | 664 | 678 | 691 | 705 | 718 | 732 | 745 | 759 | 772 |
| 322 | 786 | 799 | 813 | 826 | 840 | 853 | 866 | 880 | 893 | 907 |
| 323 | 920 | 934 | 947 | 961 | 974 | 987 | *001 | *014 | *028 | *041 |
| 324 | 51 055 | 068 | 081 | 095 | 108 | 121 | 135 | 148 | 162 | 175 |
| 325 | 188 | 202 | 215 | 228 | 242 | 255 | 268 | 282 | 295 | 308 |
| 326 | 322 | 335 | 348 | 362 | 375 | 388 | 402 | 415 | 428 | 441 |
| 327 | 455 | 468 | 481 | 495 | 508 | 521 | 534 | 548 | 561 | 574 |
| 328 | 587 | 601 | 614 | 627 | 640 | 654 | 667 | 680 | 693 | 706 |
| 329 | 720 | 733 | 746 | 759 | 772 | 786 | 799 | 812 | 825 | 838 |
| 330 | 851 | 865 | 878 | 891 | 904 | 917 | 930 | 943 | 957 | 970 |
| 331 | 983 | 996 | *009 | *022 | *035 | *048 | *061 | *075 | *088 | *101 |
| 332 | 52 114 | 127 | 140 | 153 | 166 | 179 | 192 | 205 | 218 | 231 |
| 333 | 244 | 257 | 270 | 284 | 297 | 310 | 323 | 336 | 349 | 362 |
| 334 | 375 | 388 | 401 | 414 | 427 | 440 | 453 | 466 | 479 | 492 |
| 335 | 504 | 517 | 530 | 543 | 556 | 569 | 582 | 595 | 608 | 621 |
| 336 | 634 | 647 | 660 | 673 | 686 | 699 | 711 | 724 | 737 | 750 |
| 337 | 763 | 776 | 789 | 802 | 815 | 827 | 840 | 853 | 866 | 879 |
| 338 | 892 | 905 | 917 | 930 | 943 | 956 | 969 | 982 | 994 | *007 |
| 339 | 53 020 | 033 | 046 | 058 | 071 | 084 | 097 | 110 | 122 | 135 |
| 340 | 148 | 161 | 173 | 186 | 199 | 212 | 224 | 237 | 250 | 263 |
| 341 | 275 | 288 | 301 | 314 | 326 | 339 | 352 | 364 | 377 | 390 |
| 342 | 403 | 415 | 428 | 441 | 453 | 466 | 479 | 491 | 504 | 517 |
| 343 | 529 | 542 | 555 | 567 | 580 | 593 | 605 | 618 | 631 | 643 |
| 344 | 656 | 668 | 681 | 694 | 706 | 719 | 732 | 744 | 757 | 769 |
| 345 | 782 | 794 | 807 | 820 | 832 | 845 | 857 | 870 | 882 | 895 |
| 346 | 908 | 920 | 933 | 945 | 958 | 970 | 983 | 995 | *008 | *020 |
| 347 | 54 033 | 045 | 058 | 070 | 083 | 095 | 108 | 120 | 133 | 145 |
| 348 | 158 | 170 | 183 | 195 | 208 | 220 | 233 | 245 | 258 | 270 |
| 349 | 283 | 295 | 307 | 320 | 332 | 345 | 357 | 370 | 382 | 394 |

Pp. Pts.

| | 15 | 14 | 13 | 12 |
|---|---|---|---|---|
| 1 | 1.5 | 1.4 | 1.3 | 1.2 |
| 2 | 3.0 | 2.8 | 2.6 | 2.4 |
| 3 | 4.5 | 4.2 | 3.9 | 3.6 |
| 4 | 6.0 | 5.6 | 5.2 | 4.8 |
| 5 | 7.5 | 7.0 | 6.5 | 6.0 |
| 6 | 9.0 | 8.4 | 7.8 | 7.2 |
| 7 | 10.5 | 9.8 | 9.1 | 8.4 |
| 8 | 12.0 | 11.2 | 10.4 | 9.6 |
| 9 | 13.5 | 12.6 | 11.7 | 10.8 |

## TABLE II.—LOGARITHMS OF NUMBERS.

| No. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 350 | 54 407 | 419 | 432 | 444 | 456 | 469 | 481 | 494 | 506 | 518 |
| 351 | 531 | 543 | 555 | 568 | 580 | 593 | 605 | 617 | 630 | 642 |
| 352 | 654 | 667 | 679 | 691 | 704 | 716 | 728 | 741 | 753 | 765 |
| 353 | 777 | 790 | 802 | 814 | 827 | 839 | 851 | 864 | 876 | 888 |
| 354 | 900 | 913 | 925 | 937 | 949 | 962 | 974 | 986 | 998 | *011 |
| 355 | 55 023 | 035 | 047 | 060 | 072 | 084 | 096 | 108 | 121 | 133 |
| 356 | 145 | 157 | 169 | 182 | 194 | 206 | 218 | 230 | 242 | 255 |
| 357 | 267 | 279 | 291 | 303 | 315 | 328 | 340 | 352 | 364 | 376 |
| 358 | 388 | 400 | 413 | 425 | 437 | 449 | 461 | 473 | 485 | 497 |
| 359 | 509 | 522 | 534 | 546 | 558 | 570 | 582 | 594 | 606 | 618 |
| 360 | 630 | 642 | 654 | 666 | 678 | 691 | 703 | 715 | 727 | 739 |
| 361 | 751 | 763 | 775 | 787 | 799 | 811 | 823 | 835 | 847 | 859 |
| 362 | 871 | 883 | 895 | 907 | 919 | 931 | 943 | 955 | 967 | 979 |
| 363 | 991 | *003 | *015 | *027 | *038 | *050 | *062 | *074 | *086 | *098 |
| 364 | 56 110 | 122 | 134 | 146 | 158 | 170 | 182 | 194 | 205 | 217 |
| 365 | 229 | 241 | 253 | 265 | 277 | 289 | 301 | 312 | 324 | 336 |
| 366 | 348 | 360 | 372 | 384 | 396 | 407 | 419 | 431 | 443 | 455 |
| 367 | 467 | 478 | 490 | 502 | 514 | 526 | 538 | 549 | 561 | 573 |
| 368 | 585 | 597 | 608 | 620 | 632 | 644 | 656 | 667 | 679 | 691 |
| 369 | 703 | 714 | 726 | 738 | 750 | 761 | 773 | 785 | 797 | 808 |
| 370 | 820 | 832 | 844 | 855 | 867 | 879 | 891 | 902 | 914 | 926 |
| 371 | 937 | 949 | 961 | 972 | 984 | 996 | *008 | *019 | *031 | *043 |
| 372 | 57 054 | 066 | 078 | 089 | 101 | 113 | 124 | 136 | 148 | 159 |
| 373 | 171 | 183 | 194 | 206 | 217 | 229 | 241 | 252 | 264 | 276 |
| 374 | 287 | 299 | 310 | 322 | 334 | 345 | 357 | 368 | 380 | 392 |
| 375 | 403 | 415 | 426 | 438 | 449 | 461 | 473 | 484 | 496 | 507 |
| 376 | 519 | 530 | 542 | 553 | 565 | 576 | 588 | 600 | 611 | 623 |
| 377 | 634 | 646 | 657 | 669 | 680 | 692 | 703 | 715 | 726 | 738 |
| 378 | 749 | 761 | 772 | 784 | 795 | 807 | 818 | 830 | 841 | 852 |
| 379 | 864 | 875 | 887 | 898 | 910 | 921 | 933 | 944 | 955 | 967 |
| 380 | 978 | 990 | *001 | *013 | *024 | *035 | *047 | *058 | *070 | *081 |
| 381 | 58 092 | 104 | 115 | 127 | 138 | 149 | 161 | 172 | 184 | 195 |
| 382 | 206 | 218 | 229 | 240 | 252 | 263 | 274 | 286 | 297 | 309 |
| 383 | 320 | 331 | 343 | 354 | 365 | 377 | 388 | 399 | 410 | 422 |
| 384 | 433 | 444 | 456 | 467 | 478 | 490 | 501 | 512 | 524 | 535 |
| 385 | 546 | 557 | 569 | 580 | 591 | 602 | 614 | 625 | 636 | 647 |
| 386 | 659 | 670 | 681 | 692 | 704 | 715 | 726 | 737 | 749 | 760 |
| 387 | 771 | 782 | 794 | 805 | 816 | 827 | 838 | 850 | 861 | 872 |
| 388 | 883 | 894 | 906 | 917 | 928 | 939 | 950 | 961 | 973 | 984 |
| 389 | 995 | *006 | *017 | *028 | *040 | *051 | *062 | *073 | *084 | *095 |
| 390 | 59 106 | 118 | 129 | 140 | 151 | 162 | 173 | 184 | 195 | 207 |
| 391 | 218 | 229 | 240 | 251 | 262 | 273 | 284 | 295 | 306 | 318 |
| 392 | 329 | 340 | 351 | 362 | 373 | 384 | 395 | 406 | 417 | 428 |
| 393 | 439 | 450 | 461 | 472 | 483 | 494 | 506 | 517 | 528 | 539 |
| 394 | 550 | 561 | 572 | 583 | 594 | 605 | 616 | 627 | 638 | 649 |
| 395 | 660 | 671 | 682 | 693 | 704 | 715 | 726 | 737 | 748 | 759 |
| 396 | 770 | 780 | 791 | 802 | 813 | 824 | 835 | 846 | 857 | 868 |
| 397 | 879 | 890 | 901 | 912 | 923 | 934 | 945 | 956 | 966 | 977 |
| 398 | 988 | 999 | *010 | *021 | *032 | *043 | *054 | *065 | *076 | *086 |
| 399 | 60 097 | 108 | 119 | 130 | 141 | 152 | 163 | 173 | 184 | 195 |

**Pp. Pts.**

| | 13 |
|---|---|
| 1 | 1.3 |
| 2 | 2.6 |
| 3 | 3.9 |
| 4 | 5.2 |
| 5 | 6.5 |
| 6 | 7.8 |
| 7 | 9.1 |
| 8 | 10.4 |
| 9 | 11.7 |

| | 12 |
|---|---|
| 1 | 1.2 |
| 2 | 2.4 |
| 3 | 3.6 |
| 4 | 4.8 |
| 5 | 6.0 |
| 6 | 7.2 |
| 7 | 8.4 |
| 8 | 9.6 |
| 9 | 10.8 |

| | 11 |
|---|---|
| 1 | 1.1 |
| 2 | 2.2 |
| 3 | 3.3 |
| 4 | 4.4 |
| 5 | 5.5 |
| 6 | 6.6 |
| 7 | 7.7 |
| 8 | 8.8 |
| 9 | 9.9 |

| | 10 |
|---|---|
| 1 | 1.0 |
| 2 | 2.0 |
| 3 | 3.0 |
| 4 | 4.0 |
| 5 | 5.0 |
| 6 | 6.0 |
| 7 | 7.0 |
| 8 | 8.0 |
| 9 | 9.0 |

# TABLE II.—LOGARITHMS OF NUMBERS.

| No. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 400 | 60 206 | 217 | 228 | 239 | 249 | 260 | 271 | 282 | 293 | 304 |
| 401 | 314 | 325 | 336 | 347 | 358 | 369 | 379 | 390 | 401 | 412 |
| 402 | 423 | 433 | 444 | 455 | 466 | 477 | 487 | 498 | 509 | 520 |
| 403 | 531 | 541 | 552 | 563 | 574 | 584 | 595 | 606 | 617 | 627 |
| 404 | 638 | 649 | 660 | 670 | 681 | 692 | 703 | 713 | 724 | 735 |
| 405 | 746 | 756 | 767 | 778 | 788 | 799 | 810 | 821 | 831 | 842 |
| 406 | 853 | 863 | 874 | 885 | 895 | 906 | 917 | 927 | 938 | 949 |
| 407 | 959 | 970 | 981 | 991 | *002 | *013 | *023 | *034 | *045 | *055 |
| 408 | 61 066 | 077 | 087 | 098 | 109 | 119 | 130 | 140 | 151 | 162 |
| 409 | 172 | 183 | 194 | 204 | 215 | 225 | 236 | 247 | 257 | 268 |
| 410 | 278 | 289 | 300 | 310 | 321 | 331 | 342 | 352 | 363 | 374 |
| 411 | 384 | 395 | 405 | 416 | 426 | 437 | 448 | 458 | 469 | 479 |
| 412 | 490 | 500 | 511 | 521 | 532 | 542 | 553 | 563 | 574 | 584 |
| 413 | 595 | 606 | 616 | 627 | 637 | 648 | 658 | 669 | 679 | 690 |
| 414 | 700 | 711 | 721 | 731 | 742 | 752 | 763 | 773 | 784 | 794 |
| 415 | 805 | 815 | 826 | 836 | 847 | 857 | 868 | 878 | 888 | 899 |
| 416 | 909 | 920 | 930 | 941 | 951 | 962 | 972 | 982 | 993 | *003 |
| 417 | 62 014 | 024 | 034 | 045 | 055 | 066 | 076 | 086 | 097 | 107 |
| 418 | 118 | 128 | 138 | 149 | 159 | 170 | 180 | 190 | 201 | 211 |
| 419 | 221 | 232 | 242 | 252 | 263 | 273 | 284 | 294 | 304 | 315 |
| 420 | 325 | 335 | 346 | 356 | 366 | 377 | 387 | 397 | 408 | 418 |
| 421 | 428 | 439 | 449 | 459 | 469 | 480 | 490 | 500 | 511 | 521 |
| 422 | 531 | 542 | 552 | 562 | 572 | 583 | 593 | 603 | 613 | 624 |
| 423 | 634 | 644 | 655 | 665 | 675 | 685 | 696 | 706 | 716 | 726 |
| 424 | 737 | 747 | 757 | 767 | 778 | 788 | 798 | 808 | 818 | 829 |
| 425 | 839 | 849 | 859 | 870 | 880 | 890 | 900 | 910 | 921 | 931 |
| 426 | 941 | 951 | 961 | 972 | 982 | 992 | *002 | *012 | *022 | *033 |
| 427 | 63 043 | 053 | 063 | 073 | 083 | 094 | 104 | 114 | 124 | 134 |
| 428 | 144 | 155 | 165 | 175 | 185 | 195 | 205 | 215 | 225 | 236 |
| 429 | 246 | 256 | 266 | 276 | 286 | 296 | 306 | 317 | 327 | 337 |
| 430 | 347 | 357 | 367 | 377 | 387 | 397 | 407 | 417 | 428 | 438 |
| 431 | 448 | 458 | 468 | 478 | 488 | 498 | 508 | 518 | 528 | 538 |
| 432 | 548 | 558 | 568 | 579 | 589 | 599 | 609 | 619 | 629 | 639 |
| 433 | 649 | 659 | 669 | 679 | 689 | 699 | 709 | 719 | 729 | 739 |
| 434 | 749 | 759 | 769 | 779 | 789 | 799 | 809 | 819 | 829 | 839 |
| 435 | 849 | 859 | 869 | 879 | 889 | 899 | 909 | 919 | 929 | 939 |
| 436 | 949 | 959 | 969 | 979 | 988 | 998 | *008 | *018 | *028 | *038 |
| 437 | 64 048 | 058 | 068 | 078 | 088 | 098 | 108 | 118 | 128 | 137 |
| 438 | 147 | 157 | 167 | 177 | 187 | 197 | 207 | 217 | 227 | 237 |
| 439 | 246 | 256 | 266 | 276 | 286 | 296 | 306 | 316 | 326 | 335 |
| 440 | 345 | 355 | 365 | 375 | 385 | 395 | 404 | 414 | 424 | 434 |
| 441 | 444 | 454 | 464 | 473 | 483 | 493 | 503 | 513 | 523 | 532 |
| 442 | 542 | 552 | 562 | 572 | 582 | 591 | 601 | 611 | 621 | 631 |
| 443 | 640 | 650 | 660 | 670 | 680 | 689 | 699 | 709 | 719 | 729 |
| 444 | 738 | 748 | 758 | 768 | 777 | 787 | 797 | 807 | 816 | 826 |
| 445 | 836 | 846 | 856 | 865 | 875 | 885 | 895 | 904 | 914 | 924 |
| 446 | 933 | 943 | 953 | 963 | 972 | 982 | 992 | *002 | *011 | *021 |
| 447 | 65 031 | 040 | 050 | 060 | 070 | 079 | 089 | 099 | 108 | 118 |
| 448 | 128 | 137 | 147 | 157 | 167 | 176 | 186 | 196 | 205 | 215 |
| 449 | 225 | 234 | 244 | 254 | 263 | 273 | 283 | 292 | 302 | 312 |

**Pp. Pts.**

| | 11 |
|---|---|
| 1 | 1.1 |
| 2 | 2.2 |
| 3 | 3.3 |
| 4 | 4.4 |
| 5 | 5.5 |
| 6 | 6.6 |
| 7 | 7.7 |
| 8 | 8.8 |
| 9 | 9.9 |

| | 10 |
|---|---|
| 1 | 1.0 |
| 2 | 2.0 |
| 3 | 3.0 |
| 4 | 4.0 |
| 5 | 5.0 |
| 6 | 6.0 |
| 7 | 7.0 |
| 8 | 8.0 |
| 9 | 9.0 |

| | 9 |
|---|---|
| 1 | 0.9 |
| 2 | 1.8 |
| 3 | 2.7 |
| 4 | 3.6 |
| 5 | 4.5 |
| 6 | 5.4 |
| 7 | 6.3 |
| 8 | 7.2 |
| 9 | 8.1 |

## TABLE II.—LOGARITHMS OF NUMBERS.

| No. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 450 | 65 321 | 331 | 341 | 350 | 360 | 369 | 379 | 389 | 398 | 408 |
| 451 | 418 | 427 | 437 | 447 | 456 | 466 | 475 | 485 | 495 | 504 |
| 452 | 514 | 523 | 533 | 543 | 552 | 562 | 571 | 581 | 591 | 600 |
| 453 | 610 | 619 | 629 | 639 | 648 | 658 | 667 | 677 | 686 | 696 |
| 454 | 706 | 715 | 725 | 734 | 744 | 753 | 763 | 772 | 782 | 792 |
| 455 | 801 | 811 | 820 | 830 | 839 | 849 | 858 | 868 | 877 | 887 |
| 456 | 896 | 906 | 916 | 925 | 935 | 944 | 954 | 963 | 973 | 982 |
| 457 | 992 | *001 | *011 | *020 | *030 | *039 | *049 | *058 | *068 | *077 |
| 458 | 66 087 | 096 | 106 | 115 | 124 | 134 | 143 | 153 | 162 | 172 |
| 459 | 181 | 191 | 200 | 210 | 219 | 229 | 238 | 247 | 257 | 266 |
| 460 | 276 | 285 | 295 | 304 | 314 | 323 | 332 | 342 | 351 | 361 |
| 461 | 370 | 380 | 389 | 398 | 408 | 417 | 427 | 436 | 445 | 455 |
| 462 | 464 | 474 | 483 | 492 | 502 | 511 | 521 | 530 | 539 | 549 |
| 463 | 558 | 567 | 577 | 586 | 596 | 605 | 614 | 624 | 633 | 642 |
| 464 | 652 | 661 | 671 | 680 | 689 | 699 | 708 | 717 | 727 | 736 |
| 465 | 745 | 755 | 764 | 773 | 783 | 792 | 801 | 811 | 820 | 829 |
| 466 | 839 | 848 | 857 | 867 | 876 | 885 | 894 | 904 | 913 | 922 |
| 467 | 932 | 941 | 950 | 960 | 969 | 978 | 987 | 997 | *006 | *015 |
| 468 | 67 025 | 034 | 043 | 052 | 062 | 071 | 080 | 089 | 099 | 108 |
| 469 | 117 | 127 | 136 | 145 | 154 | 164 | 173 | 182 | 191 | 201 |
| 470 | 210 | 219 | 228 | 237 | 247 | 256 | 265 | 274 | 284 | 293 |
| 471 | 302 | 311 | 321 | 330 | 339 | 348 | 357 | 367 | 376 | 385 |
| 472 | 394 | 403 | 413 | 422 | 431 | 440 | 449 | 459 | 468 | 477 |
| 473 | 486 | 495 | 504 | 514 | 523 | 532 | 541 | 550 | 560 | 569 |
| 474 | 578 | 587 | 596 | 605 | 614 | 624 | 633 | 642 | 651 | 660 |
| 475 | 669 | 679 | 688 | 697 | 706 | 715 | 724 | 733 | 742 | 752 |
| 476 | 761 | 770 | 779 | 788 | 797 | 806 | 815 | 825 | 834 | 843 |
| 477 | 852 | 861 | 870 | 879 | 888 | 897 | 906 | 916 | 925 | 934 |
| 478 | 943 | 952 | 961 | 970 | 979 | 988 | 997 | *006 | *015 | *024 |
| 479 | 68 034 | 043 | 052 | 061 | 070 | 079 | 088 | 097 | 106 | 115 |
| 480 | 124 | 133 | 142 | 151 | 160 | 169 | 178 | 187 | 196 | 205 |
| 481 | 215 | 224 | 233 | 242 | 251 | 260 | 269 | 278 | 287 | 296 |
| 482 | 305 | 314 | 323 | 332 | 341 | 350 | 359 | 368 | 377 | 386 |
| 483 | 395 | 404 | 413 | 422 | 431 | 440 | 449 | 458 | 467 | 476 |
| 484 | 485 | 494 | 502 | 511 | 520 | 529 | 538 | 547 | 556 | 565 |
| 485 | 574 | 583 | 592 | 601 | 610 | 619 | 628 | 637 | 646 | 655 |
| 486 | 664 | 673 | 681 | 690 | 699 | 708 | 717 | 726 | 735 | 744 |
| 487 | 753 | 762 | 771 | 780 | 789 | 797 | 806 | 815 | 824 | 833 |
| 488 | 842 | 851 | 860 | 869 | 878 | 886 | 895 | 904 | 913 | 922 |
| 489 | 931 | 940 | 949 | 958 | 966 | 975 | 984 | 993 | *002 | *011 |
| 490 | 69 020 | 028 | 037 | 046 | 055 | 064 | 073 | 082 | 090 | 099 |
| 491 | 108 | 117 | 126 | 135 | 144 | 152 | 161 | 170 | 179 | 188 |
| 492 | 197 | 205 | 214 | 223 | 232 | 241 | 249 | 258 | 267 | 276 |
| 493 | 285 | 294 | 302 | 311 | 320 | 329 | 338 | 346 | 355 | 364 |
| 494 | 373 | 381 | 390 | 399 | 408 | 417 | 425 | 434 | 443 | 452 |
| 495 | 461 | 469 | 478 | 487 | 496 | 504 | 513 | 522 | 531 | 539 |
| 496 | 548 | 557 | 566 | 574 | 583 | 592 | 601 | 609 | 618 | 627 |
| 497 | 636 | 644 | 653 | 662 | 671 | 679 | 688 | 697 | 705 | 714 |
| 498 | 723 | 732 | 740 | 749 | 758 | 767 | 775 | 784 | 793 | 801 |
| 499 | 810 | 819 | 827 | 836 | 845 | 854 | 862 | 871 | 880 | 888 |

**Pp. Pts.**

| | 10 |
|---|---|
| 1 | 1.0 |
| 2 | 2.0 |
| 3 | 3.0 |
| 4 | 4.0 |
| 5 | 5.0 |
| 6 | 6.0 |
| 7 | 7.0 |
| 8 | 8.0 |
| 9 | 9.0 |

| | 9 |
|---|---|
| 1 | 0.9 |
| 2 | 1.8 |
| 3 | 2.7 |
| 4 | 3.6 |
| 5 | 4.5 |
| 6 | 5.4 |
| 7 | 6.3 |
| 8 | 7.2 |
| 9 | 8.1 |

| | 8 |
|---|---|
| 1 | 0.8 |
| 2 | 1.6 |
| 3 | 2.4 |
| 4 | 3.2 |
| 5 | 4.0 |
| 6 | 4.8 |
| 7 | 5.6 |
| 8 | 6.4 |
| 9 | 7.2 |

## TABLE II.—LOGARITHMS OF NUMBERS.

| No. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Pp. Pts. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 500 | 69 897 | 906 | 914 | 923 | 932 | 940 | 949 | 958 | 966 | 975 | |
| 501 | 984 | 992 | *001 | *010 | *018 | *027 | *036 | *044 | *053 | *062 | |
| 502 | 70 070 | 079 | 088 | 096 | 105 | 114 | 122 | 131 | 140 | 148 | |
| 503 | 157 | 165 | 174 | 183 | 191 | 200 | 209 | 217 | 226 | 234 | |
| 504 | 243 | 252 | 260 | 269 | 278 | 286 | 295 | 303 | 312 | 321 | |
| 505 | 329 | 338 | 346 | 355 | 364 | 372 | 381 | 389 | 398 | 406 | |
| 506 | 415 | 424 | 432 | 441 | 449 | 458 | 467 | 475 | 484 | 492 | |
| 507 | 501 | 509 | 518 | 526 | 535 | 544 | 552 | 561 | 569 | 578 | |
| 508 | 586 | 595 | 603 | 612 | 621 | 629 | 638 | 646 | 655 | 663 | |
| 509 | 672 | 680 | 689 | 697 | 706 | 714 | 723 | 731 | 740 | 749 | |
| 510 | 757 | 766 | 774 | 783 | 791 | 800 | 808 | 817 | 825 | 834 | |
| 511 | 842 | 851 | 859 | 868 | 876 | 885 | 893 | 902 | 910 | 919 | |
| 512 | 927 | 935 | 944 | 952 | 961 | 969 | 978 | 986 | 995 | *003 | |
| 513 | 71 012 | 020 | 029 | 037 | 046 | 054 | 063 | 071 | 079 | 088 | |
| 514 | 096 | 105 | 113 | 122 | 130 | 139 | 147 | 155 | 164 | 172 | |
| 515 | 181 | 189 | 198 | 206 | 214 | 223 | 231 | 240 | 248 | 257 | |
| 516 | 265 | 273 | 282 | 290 | 299 | 307 | 315 | 324 | 332 | 341 | |
| 517 | 349 | 357 | 366 | 374 | 383 | 391 | 399 | 408 | 416 | 425 | |
| 518 | 433 | 441 | 450 | 458 | 466 | 475 | 483 | 492 | 500 | 508 | |
| 519 | 517 | 525 | 533 | 542 | 550 | 559 | 567 | 575 | 584 | 592 | |
| 520 | 600 | 609 | 617 | 625 | 634 | 642 | 650 | 659 | 667 | 675 | |
| 521 | 684 | 692 | 700 | 709 | 717 | 725 | 734 | 742 | 750 | 759 | |
| 522 | 767 | 775 | 784 | 792 | 800 | 809 | 817 | 825 | 834 | 842 | |
| 523 | 850 | 858 | 867 | 875 | 883 | 892 | 900 | 908 | 917 | 925 | |
| 524 | 933 | 941 | 950 | 958 | 966 | 975 | 983 | 991 | 999 | *008 | |
| 525 | 72 016 | 024 | 032 | 041 | 049 | 057 | 066 | 074 | 082 | 090 | |
| 526 | 099 | 107 | 115 | 123 | 132 | 140 | 148 | 156 | 165 | 173 | |
| 527 | 181 | 189 | 198 | 206 | 214 | 222 | 230 | 239 | 247 | 255 | |
| 528 | 263 | 272 | 280 | 288 | 296 | 304 | 313 | 321 | 329 | 337 | |
| 529 | 346 | 354 | 362 | 370 | 378 | 387 | 395 | 403 | 411 | 419 | |
| 530 | 428 | 436 | 444 | 452 | 460 | 469 | 477 | 485 | 493 | 501 | |
| 531 | 509 | 518 | 526 | 534 | 542 | 550 | 558 | 567 | 575 | 583 | |
| 532 | 591 | 599 | 607 | 616 | 624 | 632 | 640 | 648 | 656 | 665 | |
| 533 | 673 | 681 | 689 | 697 | 705 | 713 | 722 | 730 | 738 | 746 | |
| 534 | 754 | 762 | 770 | 779 | 787 | 795 | 803 | 811 | 819 | 827 | |
| 535 | 835 | 843 | 852 | 860 | 868 | 876 | 884 | 892 | 900 | 908 | |
| 536 | 916 | 925 | 933 | 941 | 949 | 957 | 965 | 973 | 981 | 989 | |
| 537 | 997 | *006 | *014 | *022 | *030 | *038 | *046 | *054 | *062 | *070 | |
| 538 | 73 078 | 086 | 094 | 102 | 111 | 119 | 127 | 135 | 143 | 151 | |
| 539 | 159 | 167 | 175 | 183 | 191 | 199 | 207 | 215 | 223 | 231 | |
| 540 | 239 | 247 | 255 | 263 | 272 | 280 | 288 | 296 | 304 | 312 | |
| 541 | 320 | 328 | 336 | 344 | 352 | 360 | 368 | 376 | 384 | 392 | |
| 542 | 400 | 408 | 416 | 424 | 432 | 440 | 448 | 456 | 464 | 472 | |
| 543 | 480 | 488 | 496 | 504 | 512 | 520 | 528 | 536 | 544 | 552 | |
| 544 | 560 | 568 | 576 | 584 | 592 | 600 | 608 | 616 | 624 | 632 | |
| 545 | 640 | 648 | 656 | 664 | 672 | 679 | 687 | 695 | 703 | 711 | |
| 546 | 719 | 727 | 735 | 743 | 751 | 759 | 767 | 775 | 783 | 791 | |
| 547 | 799 | 807 | 815 | 823 | 830 | 838 | 846 | 854 | 862 | 870 | |
| 548 | 878 | 886 | 894 | 902 | 910 | 918 | 926 | 933 | 941 | 949 | |
| 549 | 957 | 965 | 973 | 981 | 989 | 997 | *005 | *013 | *020 | *028 | |
| | 74 | | | | | | | | | | |

Pp. Pts.

| | 9 |
|---|---|
| 1 | 0.9 |
| 2 | 1.8 |
| 3 | 2.7 |
| 4 | 3.6 |
| 5 | 4.5 |
| 6 | 5.4 |
| 7 | 6.3 |
| 8 | 7.2 |
| 9 | 8.1 |

| | 8 |
|---|---|
| 1 | 0.8 |
| 2 | 1.6 |
| 3 | 2.4 |
| 4 | 3.2 |
| 5 | 4.0 |
| 6 | 4.8 |
| 7 | 5.6 |
| 8 | 6.4 |
| 9 | 7.2 |

| | 7 |
|---|---|
| 1 | 0.7 |
| 2 | 1.4 |
| 3 | 2.1 |
| 4 | 2.8 |
| 5 | 3.5 |
| 6 | 4.2 |
| 7 | 4.9 |
| 8 | 5.6 |
| 9 | 6.3 |

## TABLE II.—LOGARITHMS OF NUMBERS.

| No. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 550 | 74 036 | 044 | 052 | 060 | 068 | 076 | 084 | 092 | 099 | 107 |
| 551 | 115 | 123 | 131 | 139 | 147 | 155 | 162 | 170 | 178 | 186 |
| 552 | 194 | 202 | 210 | 218 | 225 | 233 | 241 | 249 | 257 | 265 |
| 553 | 273 | 280 | 288 | 296 | 304 | 312 | 320 | 327 | 335 | 343 |
| 554 | 351 | 359 | 367 | 374 | 382 | 390 | 398 | 406 | 414 | 421 |
| 555 | 429 | 437 | 445 | 453 | 461 | 468 | 476 | 484 | 492 | 500 |
| 556 | 507 | 515 | 523 | 531 | 539 | 547 | 554 | 562 | 570 | 578 |
| 557 | 586 | 593 | 601 | 609 | 617 | 624 | 632 | 640 | 648 | 656 |
| 558 | 663 | 671 | 679 | 687 | 695 | 702 | 710 | 718 | 726 | 733 |
| 559 | 741 | 749 | 757 | 764 | 772 | 780 | 788 | 796 | 803 | 811 |
| 560 | 819 | 827 | 834 | 842 | 850 | 858 | 865 | 873 | 881 | 889 |
| 561 | 896 | 904 | 912 | 920 | 927 | 935 | 943 | 950 | 958 | 966 |
| 562 | 974 | 981 | 989 | 997 | *005 | *012 | *020 | *028 | *035 | *043 |
| 563 | 75 051 | 059 | 066 | 074 | 082 | 089 | 097 | 105 | 113 | 120 |
| 564 | 128 | 136 | 143 | 151 | 159 | 166 | 174 | 182 | 189 | 197 |
| 565 | 205 | 213 | 220 | 228 | 236 | 243 | 251 | 259 | 266 | 274 |
| 566 | 282 | 289 | 297 | 305 | 312 | 320 | 328 | 335 | 343 | 351 |
| 567 | 358 | 366 | 374 | 381 | 389 | 397 | 404 | 412 | 420 | 427 |
| 568 | 435 | 442 | 450 | 458 | 465 | 473 | 481 | 488 | 496 | 504 |
| 569 | 511 | 519 | 526 | 534 | 542 | 549 | 557 | 565 | 572 | 580 |
| 570 | 587 | 595 | 603 | 610 | 618 | 626 | 633 | 641 | 648 | 656 |
| 571 | 664 | 671 | 679 | 686 | 694 | 702 | 709 | 717 | 724 | 732 |
| 572 | 740 | 747 | 755 | 762 | 770 | 778 | 785 | 793 | 800 | 808 |
| 573 | 815 | 823 | 831 | 838 | 846 | 853 | 861 | 868 | 876 | 884 |
| 574 | 891 | 899 | 906 | 914 | 921 | 929 | 937 | 944 | 952 | 959 |
| 575 | 967 | 974 | 982 | 989 | 997 | *005 | *012 | *020 | *027 | *035 |
| 576 | 76 042 | 050 | 057 | 065 | 072 | 080 | 087 | 095 | 103 | 110 |
| 577 | 118 | 125 | 133 | 140 | 148 | 155 | 163 | 170 | 178 | 185 |
| 578 | 193 | 200 | 208 | 215 | 223 | 230 | 238 | 245 | 253 | 260 |
| 579 | 268 | 275 | 283 | 290 | 298 | 305 | 313 | 320 | 328 | 335 |
| 580 | 343 | 350 | 358 | 365 | 373 | 380 | 388 | 395 | 403 | 410 |
| 581 | 418 | 425 | 433 | 440 | 448 | 455 | 462 | 470 | 477 | 485 |
| 582 | 492 | 500 | 507 | 515 | 522 | 530 | 537 | 545 | 552 | 559 |
| 583 | 567 | 574 | 582 | 589 | 597 | 604 | 612 | 619 | 626 | 634 |
| 584 | 641 | 649 | 656 | 664 | 671 | 678 | 686 | 693 | 701 | 708 |
| 585 | 716 | 723 | 730 | 738 | 745 | 753 | 760 | 768 | 775 | 782 |
| 586 | 790 | 797 | 805 | 812 | 819 | 827 | 834 | 842 | 849 | 856 |
| 587 | 864 | 871 | 879 | 886 | 893 | 901 | 908 | 916 | 923 | 930 |
| 588 | 938 | 945 | 953 | 960 | 967 | 975 | 982 | 989 | 997 | *004 |
| 589 | 77 012 | 019 | 026 | 034 | 041 | 048 | 056 | 063 | 070 | 078 |
| 590 | 085 | 093 | 100 | 107 | 115 | 122 | 129 | 137 | 144 | 151 |
| 591 | 159 | 166 | 173 | 181 | 188 | 195 | 203 | 210 | 217 | 225 |
| 592 | 232 | 240 | 247 | 254 | 262 | 269 | 276 | 283 | 291 | 298 |
| 593 | 305 | 313 | 320 | 327 | 335 | 342 | 349 | 357 | 364 | 371 |
| 594 | 379 | 386 | 393 | 401 | 408 | 415 | 422 | 430 | 437 | 444 |
| 595 | 452 | 459 | 466 | 474 | 481 | 488 | 495 | 503 | 510 | 517 |
| 596 | 525 | 532 | 539 | 546 | 554 | 561 | 568 | 576 | 583 | 590 |
| 597 | 597 | 605 | 612 | 619 | 627 | 634 | 641 | 648 | 656 | 663 |
| 598 | 670 | 677 | 685 | 692 | 699 | 706 | 714 | 721 | 728 | 735 |
| 599 | 743 | 750 | 757 | 764 | 772 | 779 | 786 | 793 | 801 | 808 |

Pp. Pts.

| | 8 |
|---|---|
| 1 | 0.8 |
| 2 | 1.6 |
| 3 | 2.4 |
| 4 | 3.2 |
| 5 | 4.0 |
| 6 | 4.8 |
| 7 | 5.6 |
| 8 | 6.4 |
| 9 | 7.2 |

| | 7 |
|---|---|
| 1 | 0.7 |
| 2 | 1.4 |
| 3 | 2.1 |
| 4 | 2.8 |
| 5 | 3.5 |
| 6 | 4.2 |
| 7 | 4.9 |
| 8 | 5.6 |
| 9 | 6.3 |

# TABLE II.—LOGARITHMS OF NUMBERS.

| No. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Pp. Pts. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 500 | 69 897 | 906 | 914 | 923 | 932 | 940 | 949 | 958 | 966 | 975 | |
| 501 | 984 | 992 | *001 | *010 | *018 | *027 | *036 | *044 | *053 | *062 | |
| 502 | 70 070 | 079 | 088 | 096 | 105 | 114 | 122 | 131 | 140 | 148 | |
| 503 | 157 | 165 | 174 | 183 | 191 | 200 | 209 | 217 | 226 | 234 | |
| 504 | 243 | 252 | 260 | 269 | 278 | 286 | 295 | 303 | 312 | 321 | |
| 505 | 329 | 338 | 346 | 355 | 364 | 372 | 381 | 389 | 398 | 406 | |
| 506 | 415 | 424 | 432 | 441 | 449 | 458 | 467 | 475 | 484 | 492 | |
| 507 | 501 | 509 | 518 | 526 | 535 | 544 | 552 | 561 | 569 | 578 | |
| 508 | 586 | 595 | 603 | 612 | 621 | 629 | 638 | 646 | 655 | 663 | 9 |
| 509 | 672 | 680 | 689 | 697 | 706 | 714 | 723 | 731 | 740 | 749 | 1 0.9 |
| 510 | 757 | 766 | 774 | 783 | 791 | 800 | 808 | 817 | 825 | 834 | 2 1.8 |
| 511 | 842 | 851 | 859 | 868 | 876 | 885 | 893 | 902 | 910 | 919 | 3 2.7 |
| 512 | 927 | 935 | 944 | 952 | 961 | 969 | 978 | 986 | 995 | *003 | 4 3.6 |
| 513 | 71 012 | 020 | 029 | 037 | 046 | 054 | 063 | 071 | 079 | 088 | 5 4.5 |
| 514 | 096 | 105 | 113 | 122 | 130 | 139 | 147 | 155 | 164 | 172 | 6 5.4 |
| 515 | 181 | 189 | 198 | 206 | 214 | 223 | 231 | 240 | 248 | 257 | 7 6.3 |
| 516 | 265 | 273 | 282 | 290 | 299 | 307 | 315 | 324 | 332 | 341 | 8 7.2 |
| 517 | 349 | 357 | 366 | 374 | 383 | 391 | 399 | 408 | 416 | 425 | 9 8.1 |
| 518 | 433 | 441 | 450 | 458 | 466 | 475 | 483 | 492 | 500 | 508 | |
| 519 | 517 | 525 | 533 | 542 | 550 | 559 | 567 | 575 | 584 | 592 | |
| 520 | 600 | 609 | 617 | 625 | 634 | 642 | 650 | 659 | 667 | 675 | |
| 521 | 684 | 692 | 700 | 709 | 717 | 725 | 734 | 742 | 750 | 759 | 8 |
| 522 | 767 | 775 | 784 | 792 | 800 | 809 | 817 | 825 | 834 | 842 | 1 0.8 |
| 523 | 850 | 858 | 867 | 875 | 883 | 892 | 900 | 908 | 917 | 925 | 2 1.6 |
| 524 | 933 | 941 | 950 | 958 | 966 | 975 | 983 | 991 | 999 | *008 | 3 2.4 |
| 525 | 72 016 | 024 | 032 | 041 | 049 | 057 | 066 | 074 | 082 | 090 | 4 3.2 |
| 526 | 099 | 107 | 115 | 123 | 132 | 140 | 148 | 156 | 165 | 173 | 5 4.0 |
| 527 | 181 | 189 | 198 | 206 | 214 | 222 | 230 | 239 | 247 | 255 | 6 4.8 |
| 528 | 263 | 272 | 280 | 288 | 296 | 304 | 313 | 321 | 329 | 337 | 7 5.6 |
| 529 | 346 | 354 | 362 | 370 | 378 | 387 | 395 | 403 | 411 | 419 | 8 6.4 |
| 530 | 428 | 436 | 444 | 452 | 460 | 469 | 477 | 485 | 493 | 501 | 9 7.2 |
| 531 | 509 | 518 | 526 | 534 | 542 | 550 | 558 | 567 | 575 | 583 | |
| 532 | 591 | 599 | 607 | 616 | 624 | 632 | 640 | 648 | 656 | 665 | |
| 533 | 673 | 681 | 689 | 697 | 705 | 713 | 722 | 730 | 738 | 746 | |
| 534 | 754 | 762 | 770 | 779 | 787 | 795 | 803 | 811 | 819 | 827 | |
| 535 | 835 | 843 | 852 | 860 | 868 | 876 | 884 | 892 | 900 | 908 | |
| 536 | 916 | 925 | 933 | 941 | 949 | 957 | 965 | 973 | 981 | 989 | 7 |
| 537 | 997 | *006 | *014 | *022 | *030 | *038 | *046 | *054 | *062 | *070 | 1 0.7 |
| 538 | 73 078 | 086 | 094 | 102 | 111 | 119 | 127 | 135 | 143 | 151 | 2 1.4 |
| 539 | 159 | 167 | 175 | 183 | 191 | 199 | 207 | 215 | 223 | 231 | 3 2.1 |
| 540 | 239 | 247 | 255 | 263 | 272 | 280 | 288 | 296 | 304 | 312 | 4 2.8 |
| 541 | 320 | 328 | 336 | 344 | 352 | 360 | 368 | 376 | 384 | 392 | 5 3.5 |
| 542 | 400 | 408 | 416 | 424 | 432 | 440 | 448 | 456 | 464 | 472 | 6 4.2 |
| 543 | 480 | 488 | 496 | 504 | 512 | 520 | 528 | 536 | 544 | 552 | 7 4.9 |
| 544 | 560 | 568 | 576 | 584 | 592 | 600 | 608 | 616 | 624 | 632 | 8 5.6 |
| 545 | 640 | 648 | 656 | 664 | 672 | 679 | 687 | 695 | 703 | 711 | 9 6.3 |
| 546 | 719 | 727 | 735 | 743 | 751 | 759 | 767 | 775 | 783 | 791 | |
| 547 | 799 | 807 | 815 | 823 | 830 | 838 | 846 | 854 | 862 | 870 | |
| 548 | 878 | 886 | 894 | 902 | 910 | 918 | 926 | 933 | 941 | 949 | |
| 549 | 957 | 965 | 973 | 981 | 989 | 997 | *005 | *013 | *020 | *028 | |
| | 74 | | | | | | | | | | |

## TABLE II.—LOGARITHMS OF NUMBERS.

| No. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 550 | 74 036 | 044 | 052 | 060 | 068 | 076 | 084 | 092 | 099 | 107 |
| 551 | 115 | 123 | 131 | 139 | 147 | 155 | 162 | 170 | 178 | 186 |
| 552 | 194 | 202 | 210 | 218 | 225 | 233 | 241 | 249 | 257 | 265 |
| 553 | 273 | 280 | 288 | 296 | 304 | 312 | 320 | 327 | 335 | 343 |
| 554 | 351 | 359 | 367 | 374 | 382 | 390 | 398 | 406 | 414 | 421 |
| 555 | 429 | 437 | 445 | 453 | 461 | 468 | 476 | 484 | 492 | 500 |
| 556 | 507 | 515 | 523 | 531 | 539 | 547 | 554 | 562 | 570 | 578 |
| 557 | 586 | 593 | 601 | 609 | 617 | 624 | 632 | 640 | 648 | 656 |
| 558 | 663 | 671 | 679 | 687 | 695 | 702 | 710 | 718 | 726 | 733 |
| 559 | 741 | 749 | 757 | 764 | 772 | 780 | 788 | 796 | 803 | 811 |
| 560 | 819 | 827 | 834 | 842 | 850 | 858 | 865 | 873 | 881 | 889 |
| 561 | 896 | 904 | 912 | 920 | 927 | 935 | 943 | 950 | 958 | 966 |
| 562 | 974 | 981 | 989 | 997 | *005 | *012 | *020 | *028 | *035 | *043 |
| 563 | 75 051 | 059 | 066 | 074 | 082 | 089 | 097 | 105 | 113 | 120 |
| 564 | 128 | 136 | 143 | 151 | 159 | 166 | 174 | 182 | 189 | 197 |
| 565 | 205 | 213 | 220 | 228 | 236 | 243 | 251 | 259 | 266 | 274 |
| 566 | 282 | 289 | 297 | 305 | 312 | 320 | 328 | 335 | 343 | 351 |
| 567 | 358 | 366 | 374 | 381 | 389 | 397 | 404 | 412 | 420 | 427 |
| 568 | 435 | 442 | 450 | 458 | 465 | 473 | 481 | 488 | 496 | 504 |
| 569 | 511 | 519 | 526 | 534 | 542 | 549 | 557 | 565 | 572 | 580 |
| 570 | 587 | 595 | 603 | 610 | 618 | 626 | 633 | 641 | 648 | 656 |
| 571 | 664 | 671 | 679 | 686 | 694 | 702 | 709 | 717 | 724 | 732 |
| 572 | 740 | 747 | 755 | 762 | 770 | 778 | 785 | 793 | 800 | 808 |
| 573 | 815 | 823 | 831 | 838 | 846 | 853 | 861 | 868 | 876 | 884 |
| 574 | 891 | 899 | 906 | 914 | 921 | 929 | 937 | 944 | 952 | 959 |
| 575 | 967 | 974 | 982 | 989 | 997 | *005 | *012 | *020 | *027 | *035 |
| 576 | 76 042 | 050 | 057 | 065 | 072 | 080 | 087 | 095 | 103 | 110 |
| 577 | 118 | 125 | 133 | 140 | 148 | 155 | 163 | 170 | 178 | 185 |
| 578 | 193 | 200 | 208 | 215 | 223 | 230 | 238 | 245 | 253 | 260 |
| 579 | 268 | 275 | 283 | 290 | 298 | 305 | 313 | 320 | 328 | 335 |
| 580 | 343 | 350 | 358 | 365 | 373 | 380 | 388 | 395 | 403 | 410 |
| 581 | 418 | 425 | 433 | 440 | 448 | 455 | 462 | 470 | 477 | 485 |
| 582 | 492 | 500 | 507 | 515 | 522 | 530 | 537 | 545 | 552 | 559 |
| 583 | 567 | 574 | 582 | 589 | 597 | 604 | 612 | 619 | 626 | 634 |
| 584 | 641 | 649 | 656 | 664 | 671 | 678 | 686 | 693 | 701 | 708 |
| 585 | 716 | 723 | 730 | 738 | 745 | 753 | 760 | 768 | 775 | 782 |
| 586 | 790 | 797 | 805 | 812 | 819 | 827 | 834 | 842 | 849 | 856 |
| 587 | 864 | 871 | 879 | 886 | 893 | 901 | 908 | 916 | 923 | 930 |
| 588 | 938 | 945 | 953 | 960 | 967 | 975 | 982 | 989 | 997 | *004 |
| 589 | 77 012 | 019 | 026 | 034 | 041 | 048 | 056 | 063 | 070 | 078 |
| 590 | 085 | 093 | 100 | 107 | 115 | 122 | 129 | 137 | 144 | 151 |
| 591 | 159 | 166 | 173 | 181 | 188 | 195 | 203 | 210 | 217 | 225 |
| 592 | 232 | 240 | 247 | 254 | 262 | 269 | 276 | 283 | 291 | 298 |
| 593 | 305 | 313 | 320 | 327 | 335 | 342 | 349 | 357 | 364 | 371 |
| 594 | 379 | 386 | 393 | 401 | 408 | 415 | 422 | 430 | 437 | 444 |
| 595 | 452 | 459 | 466 | 474 | 481 | 488 | 495 | 503 | 510 | 517 |
| 596 | 525 | 532 | 539 | 546 | 554 | 561 | 568 | 576 | 583 | 590 |
| 597 | 597 | 605 | 612 | 619 | 627 | 634 | 641 | 648 | 656 | 663 |
| 598 | 670 | 677 | 685 | 692 | 699 | 706 | 714 | 721 | 728 | 735 |
| 599 | 743 | 750 | 757 | 764 | 772 | 779 | 786 | 793 | 801 | 808 |

Pp. Pts.

| | 8 |
|---|---|
| 1 | 0.8 |
| 2 | 1.6 |
| 3 | 2.4 |
| 4 | 3.2 |
| 5 | 4.0 |
| 6 | 4.8 |
| 7 | 5.6 |
| 8 | 6.4 |
| 9 | 7.2 |

| | 7 |
|---|---|
| 1 | 0.7 |
| 2 | 1.4 |
| 3 | 2.1 |
| 4 | 2.8 |
| 5 | 3.5 |
| 6 | 4.2 |
| 7 | 4.9 |
| 8 | 5.6 |
| 9 | 6.3 |

# TABLE II.—LOGARITHMS OF NUMBERS.

| No. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Pp. Pts. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 600 | 77 815 | 822 | 830 | 837 | 844 | 851 | 859 | 866 | 873 | 880 | |
| 601 | 887 | 895 | 902 | 909 | 916 | 924 | 931 | 938 | 945 | 952 | |
| 602 | 960 | 967 | 974 | 981 | 988 | 996 | *003 | *010 | *017 | *025 | |
| 603 | 78 032 | 039 | 046 | 053 | 061 | 068 | 075 | 082 | 089 | 097 | |
| 604 | 104 | 111 | 118 | 125 | 132 | 140 | 147 | 154 | 161 | 168 | |
| 605 | 176 | 183 | 190 | 197 | 204 | 211 | 219 | 226 | 233 | 240 | |
| 606 | 247 | 254 | 262 | 269 | 276 | 283 | 290 | 297 | 305 | 312 | |
| 607 | 319 | 326 | 333 | 340 | 347 | 355 | 362 | 369 | 376 | 383 | |
| 608 | 390 | 398 | 405 | 412 | 419 | 426 | 433 | 440 | 447 | 455 | 8<br>1 0.8<br>2 1.6<br>3 2.4<br>4 3.2<br>5 4.0<br>6 4.8<br>7 5.6<br>8 6.4<br>9 7.2 |
| 609 | 462 | 469 | 476 | 483 | 490 | 497 | 504 | 512 | 519 | 526 | |
| 610 | 533 | 540 | 547 | 554 | 561 | 569 | 576 | 583 | 590 | 597 | |
| 611 | 604 | 611 | 618 | 625 | 633 | 640 | 647 | 654 | 661 | 668 | |
| 612 | 675 | 682 | 689 | 696 | 704 | 711 | 718 | 725 | 732 | 739 | |
| 613 | 746 | 753 | 760 | 767 | 774 | 781 | 789 | 796 | 803 | 810 | |
| 614 | 817 | 824 | 831 | 838 | 845 | 852 | 859 | 866 | 873 | 880 | |
| 615 | 888 | 895 | 902 | 909 | 916 | 923 | 930 | 937 | 944 | 951 | |
| 616 | 958 | 965 | 972 | 979 | 986 | 993 | *000 | *007 | *014 | *021 | |
| 617 | 79 029 | 036 | 043 | 050 | 057 | 064 | 071 | 078 | 085 | 092 | |
| 618 | 099 | 106 | 113 | 120 | 127 | 134 | 141 | 148 | 155 | 162 | |
| 619 | 169 | 176 | 183 | 190 | 197 | 204 | 211 | 218 | 225 | 232 | |
| 620 | 239 | 246 | 253 | 260 | 267 | 274 | 281 | 288 | 295 | 302 | |
| 621 | 309 | 316 | 323 | 330 | 337 | 344 | 351 | 358 | 365 | 372 | |
| 622 | 379 | 386 | 393 | 400 | 407 | 414 | 421 | 428 | 435 | 442 | 7<br>1 0.7<br>2 1.4<br>3 2.1<br>4 2.8<br>5 3.5<br>6 4.2<br>7 4.9<br>8 5.6<br>9 6.3 |
| 623 | 449 | 456 | 463 | 470 | 477 | 484 | 491 | 498 | 505 | 511 | |
| 624 | 518 | 525 | 532 | 539 | 546 | 553 | 560 | 567 | 574 | 581 | |
| 625 | 588 | 595 | 602 | 609 | 616 | 623 | 630 | 637 | 644 | 650 | |
| 626 | 657 | 664 | 671 | 678 | 685 | 692 | 699 | 706 | 713 | 720 | |
| 627 | 727 | 734 | 741 | 748 | 754 | 761 | 768 | 775 | 782 | 789 | |
| 628 | 796 | 803 | 810 | 817 | 824 | 831 | 837 | 844 | 851 | 858 | |
| 629 | 865 | 872 | 879 | 886 | 893 | 900 | 906 | 913 | 920 | 927 | |
| 630 | 934 | 941 | 948 | 955 | 962 | 969 | 975 | 982 | 989 | 996 | |
| 631 | 80 003 | 010 | 017 | 024 | 030 | 037 | 044 | 051 | 058 | 065 | |
| 632 | 072 | 079 | 085 | 092 | 099 | 106 | 113 | 120 | 127 | 134 | |
| 633 | 140 | 147 | 154 | 161 | 168 | 175 | 182 | 188 | 195 | 202 | |
| 634 | 209 | 216 | 223 | 229 | 236 | 243 | 250 | 257 | 264 | 271 | |
| 635 | 277 | 284 | 291 | 298 | 305 | 312 | 318 | 325 | 332 | 339 | |
| 636 | 346 | 353 | 359 | 366 | 373 | 380 | 387 | 393 | 400 | 407 | 6<br>1 0.6<br>2 1.2<br>3 1.8<br>4 2.4<br>5 3.0<br>6 3.6<br>7 4.2<br>8 4.8<br>9 5.4 |
| 637 | 414 | 421 | 428 | 434 | 441 | 448 | 455 | 462 | 468 | 475 | |
| 638 | 482 | 489 | 496 | 502 | 509 | 516 | 523 | 530 | 536 | 543 | |
| 639 | 550 | 557 | 564 | 570 | 577 | 584 | 591 | 598 | 604 | 611 | |
| 640 | 618 | 625 | 632 | 638 | 645 | 652 | 659 | 665 | 672 | 679 | |
| 641 | 686 | 693 | 699 | 706 | 713 | 720 | 726 | 733 | 740 | 747 | |
| 642 | 754 | 760 | 767 | 774 | 781 | 787 | 794 | 801 | 808 | 814 | |
| 643 | 821 | 828 | 835 | 841 | 848 | 855 | 862 | 868 | 875 | 882 | |
| 644 | 889 | 895 | 902 | 909 | 916 | 922 | 929 | 936 | 943 | 949 | |
| 645 | 956 | 963 | 969 | 976 | 983 | 990 | 996 | *003 | *010 | *017 | |
| 646 | 81 023 | 030 | 037 | 043 | 050 | 057 | 064 | 070 | 077 | 084 | |
| 647 | 090 | 097 | 104 | 111 | 117 | 124 | 131 | 137 | 144 | 151 | |
| 648 | 158 | 164 | 171 | 178 | 184 | 191 | 198 | 204 | 211 | 218 | |
| 649 | 224 | 231 | 238 | 245 | 251 | 258 | 265 | 271 | 278 | 285 | |

## TABLE II.—LOGARITHMS OF NUMBERS.

| No. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Pp. Pts. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 650 | 81 291 | 298 | 305 | 311 | 318 | 325 | 331 | 338 | 345 | 351 | |
| 651 | 358 | 365 | 371 | 378 | 385 | 391 | 398 | 405 | 411 | 418 | |
| 652 | 425 | 431 | 438 | 445 | 451 | 458 | 465 | 471 | 478 | 485 | |
| 653 | 491 | 498 | 505 | 511 | 518 | 525 | 531 | 538 | 544 | 551 | |
| 654 | 558 | 564 | 571 | 578 | 584 | 591 | 598 | 604 | 611 | 617 | |
| 655 | 624 | 631 | 637 | 644 | 651 | 657 | 664 | 671 | 677 | 684 | |
| 656 | 690 | 697 | 704 | 710 | 717 | 723 | 730 | 737 | 743 | 750 | |
| 657 | 757 | 763 | 770 | 776 | 783 | 790 | 796 | 803 | 809 | 816 | |
| 658 | 823 | 829 | 836 | 842 | 849 | 856 | 862 | 869 | 875 | 882 | |
| 659 | 889 | 895 | 902 | 908 | 915 | 921 | 928 | 935 | 941 | 948 | |
| 660 | 954 | 961 | 968 | 974 | 981 | 987 | 994 | *000 | *007 | *014 | |
| 661 | 82 020 | 027 | 033 | 040 | 046 | 053 | 060 | 066 | 073 | 079 | |
| 662 | 086 | 092 | 099 | 105 | 112 | 119 | 125 | 132 | 138 | 145 | |
| 663 | 151 | 158 | 164 | 171 | 178 | 184 | 191 | 197 | 204 | 210 | 7 |
| 664 | 217 | 223 | 230 | 236 | 243 | 249 | 256 | 263 | 269 | 276 | 1 0.7 |
| 665 | 282 | 289 | 295 | 302 | 308 | 315 | 321 | 328 | 334 | 341 | 2 1.4 |
| 666 | 347 | 354 | 360 | 367 | 373 | 380 | 387 | 393 | 400 | 406 | 3 2.1 |
| 667 | 413 | 419 | 426 | 432 | 439 | 445 | 452 | 458 | 465 | 471 | 4 2.8 |
| 668 | 478 | 484 | 491 | 497 | 504 | 510 | 517 | 523 | 530 | 536 | 5 3.5 |
| 669 | 543 | 549 | 556 | 562 | 569 | 575 | 582 | 588 | 595 | 601 | 6 4.2 |
| 670 | 607 | 614 | 620 | 627 | 633 | 640 | 646 | 653 | 659 | 666 | 7 4.9 |
| 671 | 672 | 679 | 685 | 692 | 698 | 705 | 711 | 718 | 724 | 730 | 8 5.6 |
| 672 | 737 | 743 | 750 | 756 | 763 | 769 | 776 | 782 | 789 | 795 | 9 6.3 |
| 673 | 802 | 808 | 814 | 821 | 827 | 834 | 840 | 847 | 853 | 860 | |
| 674 | 866 | 872 | 879 | 885 | 892 | 898 | 905 | 911 | 918 | 924 | |
| 675 | 930 | 937 | 943 | 950 | 956 | 963 | 969 | 975 | 982 | 988 | |
| 676 | 995 | *001 | *008 | *014 | *020 | *027 | *033 | *040 | *046 | *052 | |
| 677 | 83 059 | 065 | 072 | 078 | 085 | 091 | 097 | 104 | 110 | 117 | |
| 678 | 123 | 129 | 136 | 142 | 149 | 155 | 161 | 168 | 174 | 181 | |
| 679 | 187 | 193 | 200 | 206 | 213 | 219 | 225 | 232 | 238 | 245 | |
| 680 | 251 | 257 | 264 | 270 | 276 | 283 | 289 | 296 | 302 | 308 | |
| 681 | 315 | 321 | 327 | 334 | 340 | 347 | 353 | 359 | 366 | 372 | 6 |
| 682 | 378 | 385 | 391 | 398 | 404 | 410 | 417 | 423 | 429 | 436 | 1 0.6 |
| 683 | 442 | 448 | 455 | 461 | 467 | 474 | 480 | 487 | 493 | 499 | 2 1.2 |
| 684 | 506 | 512 | 518 | 525 | 531 | 537 | 544 | 550 | 556 | 563 | 3 1.8 |
| 685 | 569 | 575 | 582 | 588 | 594 | 601 | 607 | 613 | 620 | 626 | 4 2.4 |
| 686 | 632 | 639 | 645 | 651 | 658 | 664 | 670 | 677 | 683 | 689 | 5 3.0 |
| 687 | 696 | 702 | 708 | 715 | 721 | 727 | 734 | 740 | 746 | 753 | 6 3.6 |
| 688 | 759 | 765 | 771 | 778 | 784 | 790 | 797 | 803 | 809 | 816 | 7 4.2 |
| 689 | 822 | 828 | 835 | 841 | 847 | 853 | 860 | 866 | 872 | 879 | 8 4.8 |
| 690 | 885 | 891 | 897 | 904 | 910 | 916 | 923 | 929 | 935 | 942 | 9 5.4 |
| 691 | 948 | 954 | 960 | 967 | 973 | 979 | 985 | 992 | 998 | *004 | |
| 692 | 84 011 | 017 | 023 | 029 | 036 | 042 | 048 | 055 | 061 | 067 | |
| 693 | 073 | 080 | 086 | 092 | 098 | 105 | 111 | 117 | 123 | 130 | |
| 694 | 136 | 142 | 148 | 155 | 161 | 167 | 173 | 180 | 186 | 192 | |
| 695 | 198 | 205 | 211 | 217 | 223 | 230 | 236 | 242 | 248 | 255 | |
| 696 | 261 | 267 | 273 | 280 | 286 | 292 | 298 | 305 | 311 | 317 | |
| 697 | 323 | 330 | 336 | 342 | 348 | 354 | 361 | 367 | 373 | 379 | |
| 698 | 386 | 392 | 398 | 404 | 410 | 417 | 423 | 429 | 435 | 442 | |
| 699 | 448 | 454 | 460 | 466 | 473 | 479 | 485 | 491 | 497 | 504 | |

# TABLE II.—LOGARITHMS OF NUMBERS.

| No. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Pp. Pts. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 700 | 84 510 | 516 | 522 | 528 | 535 | 541 | 547 | 553 | 559 | 566 | |
| 701 | 572 | 578 | 584 | 590 | 597 | 603 | 609 | 615 | 621 | 628 | |
| 702 | 634 | 640 | 646 | 652 | 658 | 665 | 671 | 677 | 683 | 689 | |
| 703 | 696 | 702 | 708 | 714 | 720 | 726 | 733 | 739 | 745 | 751 | |
| 704 | 757 | 763 | 770 | 776 | 782 | 788 | 794 | 800 | 807 | 813 | |
| 705 | 819 | 825 | 831 | 837 | 844 | 850 | 856 | 862 | 868 | 874 | |
| 706 | 880 | 887 | 893 | 899 | 905 | 911 | 917 | 924 | 930 | 936 | |
| 707 | 942 | 948 | 954 | 960 | 967 | 973 | 979 | 985 | 991 | 997 | |
| 708 | 85 003 | 009 | 016 | 022 | 028 | 034 | 040 | 046 | 052 | 058 | 7 |
| 709 | 065 | 071 | 077 | 083 | 089 | 095 | 101 | 107 | 114 | 120 | 1 0.7 |
| 710 | 126 | 132 | 138 | 144 | 150 | 156 | 163 | 169 | 175 | 181 | 2 1.4 |
| 711 | 187 | 193 | 199 | 205 | 211 | 217 | 224 | 230 | 236 | 242 | 3 2.1 |
| 712 | 248 | 254 | 260 | 266 | 272 | 278 | 285 | 291 | 297 | 303 | 4 2.8 |
| 713 | 309 | 315 | 321 | 327 | 333 | 339 | 345 | 352 | 358 | 364 | 5 3.5 |
| 714 | 370 | 376 | 382 | 388 | 394 | 400 | 406 | 412 | 418 | 425 | 6 4.2 |
| 715 | 431 | 437 | 443 | 449 | 455 | 461 | 467 | 473 | 479 | 485 | 7 4.9 |
| 716 | 491 | 497 | 503 | 509 | 516 | 522 | 528 | 534 | 540 | 546 | 8 5.6 |
| 717 | 552 | 558 | 564 | 570 | 576 | 582 | 588 | 594 | 600 | 606 | 9 6.3 |
| 718 | 612 | 618 | 625 | 631 | 637 | 643 | 649 | 655 | 661 | 667 | |
| 719 | 673 | 679 | 685 | 691 | 697 | 703 | 709 | 715 | 721 | 727 | |
| 720 | 733 | 739 | 745 | 751 | 757 | 763 | 769 | 775 | 781 | 788 | |
| 721 | 794 | 800 | 806 | 812 | 818 | 824 | 830 | 836 | 842 | 848 | 6 |
| 722 | 854 | 860 | 866 | 872 | 878 | 884 | 890 | 896 | 902 | 908 | 1 0.6 |
| 723 | 914 | 920 | 926 | 932 | 938 | 944 | 950 | 956 | 962 | 968 | 2 1.2 |
| 724 | 974 | 980 | 986 | 992 | 998 | *004 | *010 | *016 | *022 | *028 | 3 1.8 |
| 725 | 86 034 | 040 | 046 | 052 | 058 | 064 | 070 | 076 | 082 | 088 | 4 2.4 |
| 726 | 094 | 100 | 106 | 112 | 118 | 124 | 130 | 136 | 141 | 147 | 5 3.0 |
| 727 | 153 | 159 | 165 | 171 | 177 | 183 | 189 | 195 | 201 | 207 | 6 3.6 |
| 728 | 213 | 219 | 225 | 231 | 237 | 243 | 249 | 255 | 261 | 267 | 7 4.2 |
| 729 | 273 | 279 | 285 | 291 | 297 | 303 | 308 | 314 | 320 | 326 | 8 4.8 |
| 730 | 332 | 338 | 344 | 350 | 356 | 362 | 368 | 374 | 380 | 386 | 9 5.4 |
| 731 | 392 | 398 | 404 | 410 | 415 | 421 | 427 | 433 | 439 | 445 | |
| 732 | 451 | 457 | 463 | 469 | 475 | 481 | 487 | 493 | 499 | 504 | |
| 733 | 510 | 516 | 522 | 528 | 534 | 540 | 546 | 552 | 558 | 564 | |
| 734 | 570 | 576 | 581 | 587 | 593 | 599 | 605 | 611 | 617 | 623 | |
| 735 | 629 | 635 | 641 | 646 | 652 | 658 | 664 | 670 | 676 | 682 | 5 |
| 736 | 688 | 694 | 700 | 705 | 711 | 717 | 723 | 729 | 735 | 741 | 1 0.5 |
| 737 | 747 | 753 | 759 | 764 | 770 | 776 | 782 | 788 | 794 | 800 | 2 1.0 |
| 738 | 806 | 812 | 817 | 823 | 829 | 835 | 841 | 847 | 853 | 859 | 3 1.5 |
| 739 | 864 | 870 | 876 | 882 | 888 | 894 | 900 | 906 | 911 | 917 | 4 2.0 |
| 740 | 923 | 929 | 935 | 941 | 947 | 953 | 958 | 964 | 970 | 976 | 5 2.5 |
| 741 | 982 | 988 | 994 | 999 | *005 | *011 | *017 | *023 | *029 | *035 | 6 3.0 |
| 742 | 87 040 | 046 | 052 | 058 | 064 | 070 | 075 | 081 | 087 | 093 | 7 3.5 |
| 743 | 099 | 105 | 111 | 116 | 122 | 128 | 134 | 140 | 146 | 151 | 8 4.0 |
| 744 | 157 | 163 | 169 | 175 | 181 | 186 | 192 | 198 | 204 | 210 | 9 4.5 |
| 745 | 216 | 221 | 227 | 233 | 239 | 245 | 251 | 256 | 262 | 268 | |
| 746 | 274 | 280 | 286 | 291 | 297 | 303 | 309 | 315 | 320 | 326 | |
| 747 | 332 | 338 | 344 | 349 | 355 | 361 | 367 | 373 | 379 | 384 | |
| 748 | 390 | 396 | 402 | 408 | 413 | 419 | 425 | 431 | 437 | 442 | |
| 749 | 448 | 454 | 460 | 466 | 471 | 477 | 483 | 489 | 495 | 500 | |

## TABLE II.—LOGARITHMS OF NUMBERS.

| No. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 750 | 87 506 | 512 | 518 | 523 | 529 | 535 | 541 | 547 | 552 | 558 |
| 751 | 564 | 570 | 576 | 581 | 587 | 593 | 599 | 604 | 610 | 616 |
| 752 | 622 | 628 | 633 | 639 | 645 | 651 | 656 | 662 | 668 | 674 |
| 753 | 679 | 685 | 691 | 697 | 703 | 708 | 714 | 720 | 726 | 731 |
| 754 | 737 | 743 | 749 | 754 | 760 | 766 | 772 | 777 | 783 | 789 |
| 755 | 795 | 800 | 806 | 812 | 818 | 823 | 829 | 835 | 841 | 846 |
| 756 | 852 | 858 | 864 | 869 | 875 | 881 | 887 | 892 | 898 | 904 |
| 757 | 910 | 915 | 921 | 927 | 933 | 938 | 944 | 950 | 955 | 961 |
| 758 | 967 | 973 | 978 | 984 | 990 | 996 | *001 | *007 | *013 | *018 |
| 759 | 88 024 | 030 | 036 | 041 | 047 | 053 | 058 | 064 | 070 | 076 |
| 760 | 081 | 087 | 093 | 098 | 104 | 110 | 116 | 121 | 127 | 133 |
| 761 | 138 | 144 | 150 | 156 | 161 | 167 | 173 | 178 | 184 | 190 |
| 762 | 195 | 201 | 207 | 213 | 218 | 224 | 230 | 235 | 241 | 247 |
| 763 | 252 | 258 | 264 | 270 | 275 | 281 | 287 | 292 | 298 | 304 |
| 764 | 309 | 315 | 321 | 326 | 332 | 338 | 343 | 349 | 355 | 360 |
| 765 | 366 | 372 | 377 | 383 | 389 | 395 | 400 | 406 | 412 | 417 |
| 766 | 423 | 429 | 434 | 440 | 446 | 451 | 457 | 463 | 468 | 474 |
| 767 | 480 | 485 | 491 | 497 | 502 | 508 | 513 | 519 | 525 | 530 |
| 768 | 536 | 542 | 547 | 553 | 559 | 564 | 570 | 576 | 581 | 587 |
| 769 | 593 | 598 | 604 | 610 | 615 | 621 | 627 | 632 | 638 | 643 |
| 770 | 649 | 655 | 660 | 666 | 672 | 677 | 683 | 689 | 694 | 700 |
| 771 | 705 | 711 | 717 | 722 | 728 | 734 | 739 | 745 | 750 | 756 |
| 772 | 762 | 767 | 773 | 779 | 784 | 790 | 795 | 801 | 807 | 812 |
| 773 | 818 | 824 | 829 | 835 | 840 | 846 | 852 | 857 | 863 | 868 |
| 774 | 874 | 880 | 885 | 891 | 897 | 902 | 908 | 913 | 919 | 925 |
| 775 | 930 | 936 | 941 | 947 | 953 | 958 | 964 | 969 | 975 | 981 |
| 776 | 986 | 992 | 997 | *003 | *009 | *014 | *020 | *025 | *031 | *037 |
| 777 | 89 042 | 048 | 053 | 059 | 064 | 070 | 076 | 081 | 087 | 092 |
| 778 | 098 | 104 | 109 | 115 | 120 | 126 | 131 | 137 | 143 | 148 |
| 779 | 154 | 159 | 165 | 170 | 176 | 182 | 187 | 193 | 198 | 204 |
| 780 | 209 | 215 | 221 | 226 | 232 | 237 | 243 | 248 | 254 | 260 |
| 781 | 265 | 271 | 276 | 282 | 287 | 293 | 298 | 304 | 310 | 315 |
| 782 | 321 | 326 | 332 | 337 | 343 | 348 | 354 | 360 | 365 | 371 |
| 783 | 376 | 382 | 387 | 393 | 398 | 404 | 409 | 415 | 421 | 426 |
| 784 | 432 | 437 | 443 | 448 | 454 | 459 | 465 | 470 | 476 | 481 |
| 785 | 487 | 492 | 498 | 504 | 509 | 515 | 520 | 526 | 531 | 537 |
| 786 | 542 | 548 | 553 | 559 | 564 | 570 | 575 | 581 | 586 | 592 |
| 787 | 597 | 603 | 609 | 614 | 620 | 625 | 631 | 636 | 642 | 647 |
| 788 | 653 | 658 | 664 | 669 | 675 | 680 | 686 | 691 | 697 | 702 |
| 789 | 708 | 713 | 719 | 724 | 730 | 735 | 741 | 746 | 752 | 757 |
| 790 | 763 | 768 | 774 | 779 | 785 | 790 | 796 | 801 | 807 | 812 |
| 791 | 818 | 823 | 829 | 834 | 840 | 845 | 851 | 856 | 862 | 867 |
| 792 | 873 | 878 | 883 | 889 | 894 | 900 | 905 | 911 | 916 | 922 |
| 793 | 927 | 933 | 938 | 944 | 949 | 955 | 960 | 966 | 971 | 977 |
| 794 | 982 | 988 | 993 | 998 | *004 | *009 | *015 | *020 | *026 | *031 |
| 795 | 90 037 | 042 | 048 | 053 | 059 | 064 | 069 | 075 | 080 | 086 |
| 796 | 091 | 097 | 102 | 108 | 113 | 119 | 124 | 129 | 135 | 140 |
| 797 | 146 | 151 | 157 | 162 | 168 | 173 | 179 | 184 | 189 | 195 |
| 798 | 200 | 206 | 211 | 217 | 222 | 227 | 233 | 238 | 244 | 249 |
| 799 | 255 | 260 | 266 | 271 | 276 | 282 | 287 | 293 | 298 | 304 |

Pp. Pts.

| | 6 |
|---|---|
| 1 | 0.6 |
| 2 | 1.2 |
| 3 | 1.8 |
| 4 | 2.4 |
| 5 | 3.0 |
| 6 | 3.6 |
| 7 | 4.2 |
| 8 | 4.8 |
| 9 | 5.4 |

| | 5 |
|---|---|
| 1 | 0.5 |
| 2 | 1.0 |
| 3 | 1.5 |
| 4 | 2.0 |
| 5 | 2.5 |
| 6 | 3.0 |
| 7 | 3.5 |
| 8 | 4.0 |
| 9 | 4.5 |

| No. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Pp. Pts. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 800 | 90 309 | 314 | 320 | 325 | 331 | 336 | 342 | 347 | 352 | 358 | |
| 801 | 363 | 369 | 374 | 380 | 385 | 390 | 396 | 401 | 407 | 412 | |
| 802 | 417 | 423 | 428 | 434 | 439 | 445 | 450 | 455 | 461 | 466 | |
| 803 | 472 | 477 | 482 | 488 | 493 | 499 | 504 | 509 | 515 | 520 | |
| 804 | 526 | 531 | 536 | 542 | 547 | 553 | 558 | 563 | 569 | 574 | |
| 805 | 580 | 585 | 590 | 596 | 601 | 607 | 612 | 617 | 623 | 628 | |
| 806 | 634 | 639 | 644 | 650 | 655 | 660 | 666 | 671 | 677 | 682 | |
| 807 | 687 | 693 | 698 | 703 | 709 | 714 | 720 | 725 | 730 | 736 | |
| 808 | 741 | 747 | 752 | 757 | 763 | 768 | 773 | 779 | 784 | 789 | |
| 809 | 795 | 800 | 806 | 811 | 816 | 822 | 827 | 832 | 838 | 843 | |
| 810 | 849 | 854 | 859 | 865 | 870 | 875 | 881 | 886 | 891 | 897 | |
| 811 | 902 | 907 | 913 | 918 | 924 | 929 | 934 | 940 | 945 | 950 | |
| 812 | 956 | 961 | 966 | 972 | 977 | 982 | 988 | 993 | 998 | *004 | 6 |
| 813 | 91 009 | 014 | 020 | 025 | 030 | 036 | 041 | 046 | 052 | 057 | 1 0.6 |
| 814 | 062 | 068 | 073 | 078 | 084 | 089 | 094 | 100 | 105 | 110 | 2 1.2 |
| 815 | 116 | 121 | 126 | 132 | 137 | 142 | 148 | 153 | 158 | 164 | 3 1.8 |
| 816 | 169 | 174 | 180 | 185 | 190 | 196 | 201 | 206 | 212 | 217 | 4 2.4 |
| 817 | 222 | 228 | 233 | 238 | 243 | 249 | 254 | 259 | 265 | 270 | 5 3.0 |
| 818 | 275 | 281 | 286 | 291 | 297 | 302 | 307 | 312 | 318 | 323 | 6 3.6 |
| 819 | 328 | 334 | 339 | 344 | 350 | 355 | 360 | 365 | 371 | 376 | 7 4.2 |
| 820 | 381 | 387 | 392 | 397 | 403 | 408 | 413 | 418 | 424 | 429 | 8 4.8 |
| 821 | 434 | 440 | 445 | 450 | 455 | 461 | 466 | 471 | 477 | 482 | 9 5.4 |
| 822 | 487 | 492 | 498 | 503 | 508 | 514 | 519 | 524 | 529 | 535 | |
| 823 | 540 | 545 | 551 | 556 | 561 | 566 | 572 | 577 | 582 | 587 | |
| 824 | 593 | 598 | 603 | 609 | 614 | 619 | 624 | 630 | 635 | 640 | |
| 825 | 645 | 651 | 656 | 661 | 666 | 672 | 677 | 682 | 687 | 693 | |
| 826 | 698 | 703 | 709 | 714 | 719 | 724 | 730 | 735 | 740 | 745 | |
| 827 | 751 | 756 | 761 | 766 | 772 | 777 | 782 | 787 | 793 | 798 | |
| 828 | 803 | 808 | 814 | 819 | 824 | 829 | 834 | 840 | 845 | 850 | |
| 829 | 855 | 861 | 866 | 871 | 876 | 882 | 887 | 892 | 897 | 903 | |
| 830 | 908 | 913 | 918 | 924 | 929 | 934 | 939 | 944 | 950 | 955 | |
| 831 | 960 | 965 | 971 | 976 | 981 | 986 | 991 | 997 | *002 | *007 | 5 |
| 832 | 92 012 | 018 | 023 | 028 | 033 | 038 | 044 | 049 | 054 | 059 | 1 0.5 |
| 833 | 065 | 070 | 075 | 080 | 085 | 091 | 096 | 101 | 106 | 111 | 2 1.0 |
| 834 | 117 | 122 | 127 | 132 | 137 | 143 | 148 | 153 | 158 | 163 | 3 1.5 |
| 835 | 169 | 174 | 179 | 184 | 189 | 195 | 200 | 205 | 210 | 215 | 4 2.0 |
| 836 | 221 | 226 | 231 | 236 | 241 | 247 | 252 | 257 | 262 | 267 | 5 2.5 |
| 837 | 273 | 278 | 283 | 288 | 293 | 298 | 304 | 309 | 314 | 319 | 6 3.0 |
| 838 | 324 | 330 | 335 | 340 | 345 | 350 | 355 | 361 | 366 | 371 | 7 3.5 |
| 839 | 376 | 381 | 387 | 392 | 397 | 402 | 407 | 412 | 418 | 423 | 8 4.0 |
| 840 | 428 | 433 | 438 | 443 | 449 | 454 | 459 | 464 | 469 | 474 | 9 4.5 |
| 841 | 480 | 485 | 490 | 495 | 500 | 505 | 511 | 516 | 521 | 526 | |
| 842 | 531 | 536 | 542 | 547 | 552 | 557 | 562 | 567 | 572 | 578 | |
| 843 | 583 | 588 | 593 | 598 | 603 | 609 | 614 | 619 | 624 | 629 | |
| 844 | 634 | 639 | 645 | 650 | 655 | 660 | 665 | 670 | 675 | 681 | |
| 845 | 686 | 691 | 696 | 701 | 706 | 711 | 716 | 722 | 727 | 732 | |
| 846 | 737 | 742 | 747 | 752 | 758 | 763 | 768 | 773 | 778 | 783 | |
| 847 | 788 | 793 | 799 | 804 | 809 | 814 | 819 | 824 | 829 | 834 | |
| 848 | 840 | 845 | 850 | 855 | 860 | 865 | 870 | 875 | 881 | 886 | |
| 849 | 891 | 896 | 901 | 906 | 911 | 916 | 921 | 927 | 932 | 937 | |

## TABLE II.—LOGARITHMS OF NUMBERS.

| No. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Pp. Pts. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 850 | 92 942 | 947 | 952 | 957 | 962 | 967 | 973 | 978 | 983 | 988 | |
| 851 | 993 | 998 | *003 | *008 | *013 | *018 | *024 | *029 | *034 | *039 | |
| 852 | 93 044 | 049 | 054 | 059 | 064 | 069 | 075 | 080 | 085 | 090 | |
| 853 | 095 | 100 | 105 | 110 | 115 | 120 | 125 | 131 | 136 | 141 | |
| 854 | 146 | 151 | 156 | 161 | 166 | 171 | 176 | 181 | 186 | 192 | |
| 855 | 197 | 202 | 207 | 212 | 217 | 222 | 227 | 232 | 237 | 242 | |
| 856 | 247 | 252 | 258 | 263 | 268 | 273 | 278 | 283 | 288 | 293 | |
| 857 | 298 | 303 | 308 | 313 | 318 | 323 | 328 | 334 | 339 | 344 | |
| 858 | 349 | 354 | 359 | 364 | 369 | 374 | 379 | 384 | 389 | 394 | 6<br>1 0.6<br>2 1.2<br>3 1.8<br>4 2.4<br>5 3.0<br>6 3.6<br>7 4.2<br>8 4.8<br>9 5.4 |
| 859 | 399 | 404 | 409 | 414 | 420 | 425 | 430 | 435 | 440 | 445 | |
| 860 | 450 | 455 | 460 | 465 | 470 | 475 | 480 | 485 | 490 | 495 | |
| 861 | 500 | 505 | 510 | 515 | 520 | 526 | 531 | 536 | 541 | 546 | |
| 862 | 551 | 556 | 561 | 566 | 571 | 576 | 581 | 586 | 591 | 596 | |
| 863 | 601 | 606 | 611 | 616 | 621 | 626 | 631 | 636 | 641 | 646 | |
| 864 | 651 | 656 | 661 | 666 | 671 | 676 | 682 | 687 | 692 | 697 | |
| 865 | 702 | 707 | 712 | 717 | 722 | 727 | 732 | 737 | 742 | 747 | |
| 866 | 752 | 757 | 762 | 767 | 772 | 777 | 782 | 787 | 792 | 797 | |
| 867 | 802 | 807 | 812 | 817 | 822 | 827 | 832 | 837 | 842 | 847 | |
| 868 | 852 | 857 | 862 | 867 | 872 | 877 | 882 | 887 | 892 | 897 | |
| 869 | 902 | 907 | 912 | 917 | 922 | 927 | 932 | 937 | 942 | 947 | |
| 870 | 952 | 957 | 962 | 967 | 972 | 977 | 982 | 987 | 992 | 997 | |
| 871 | 94 002 | 007 | 012 | 017 | 022 | 027 | 032 | 037 | 042 | 047 | |
| 872 | 052 | 057 | 062 | 067 | 072 | 077 | 082 | 086 | 091 | 096 | 5<br>1 0.5<br>2 1.0<br>3 1.5<br>4 2.0<br>5 2.5<br>6 3.0<br>7 3.5<br>8 4.0<br>9 4.5 |
| 873 | 101 | 106 | 111 | 116 | 121 | 126 | 131 | 136 | 141 | 146 | |
| 874 | 151 | 156 | 161 | 166 | 171 | 176 | 181 | 186 | 191 | 196 | |
| 875 | 201 | 206 | 211 | 216 | 221 | 226 | 231 | 236 | 240 | 245 | |
| 876 | 250 | 255 | 260 | 265 | 270 | 275 | 280 | 285 | 290 | 295 | |
| 877 | 300 | 305 | 310 | 315 | 320 | 325 | 330 | 335 | 340 | 345 | |
| 878 | 349 | 354 | 359 | 364 | 369 | 374 | 379 | 384 | 389 | 394 | |
| 879 | 399 | 404 | 409 | 414 | 419 | 424 | 429 | 433 | 438 | 443 | |
| 880 | 448 | 453 | 458 | 463 | 468 | 473 | 478 | 483 | 488 | 493 | |
| 881 | 498 | 503 | 507 | 512 | 517 | 522 | 527 | 532 | 537 | 542 | |
| 882 | 547 | 552 | 557 | 562 | 567 | 571 | 576 | 581 | 586 | 591 | |
| 883 | 596 | 601 | 606 | 611 | 616 | 621 | 626 | 630 | 635 | 640 | |
| 884 | 645 | 650 | 655 | 660 | 665 | 670 | 675 | 680 | 685 | 689 | |
| 885 | 694 | 699 | 704 | 709 | 714 | 719 | 724 | 729 | 734 | 738 | |
| 886 | 743 | 748 | 753 | 758 | 763 | 768 | 773 | 778 | 783 | 787 | 4<br>1 0.4<br>2 0.8<br>3 1.2<br>4 1.6<br>5 2.0<br>6 2.4<br>7 2.8<br>8 3.2<br>9 3.6 |
| 887 | 792 | 797 | 802 | 807 | 812 | 817 | 822 | 827 | 832 | 836 | |
| 888 | 841 | 846 | 851 | 856 | 861 | 866 | 871 | 876 | 880 | 885 | |
| 889 | 890 | 895 | 900 | 905 | 910 | 915 | 919 | 924 | 929 | 934 | |
| 890 | 939 | 944 | 949 | 954 | 959 | 963 | 968 | 973 | 978 | 983 | |
| 891 | 988 | 993 | 998 | *002 | *007 | *012 | *017 | *022 | *027 | *032 | |
| 892 | 95 036 | 041 | 046 | 051 | 056 | 061 | 066 | 071 | 075 | 080 | |
| 893 | 085 | 090 | 095 | 100 | 105 | 109 | 114 | 119 | 124 | 129 | |
| 894 | 134 | 139 | 143 | 148 | 153 | 158 | 163 | 168 | 173 | 177 | |
| 895 | 182 | 187 | 192 | 197 | 202 | 207 | 211 | 216 | 221 | 226 | |
| 896 | 231 | 236 | 240 | 245 | 250 | 255 | 260 | 265 | 270 | 274 | |
| 897 | 279 | 284 | 289 | 294 | 299 | 303 | 308 | 313 | 318 | 323 | |
| 898 | 328 | 332 | 337 | 342 | 347 | 352 | 357 | 361 | 366 | 371 | |
| 899 | 376 | 381 | 386 | 390 | 395 | 400 | 405 | 410 | 415 | 419 | |

## TABLE II.—LOGARITHMS OF NUMBERS.

| No. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Pp. Pts. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 900 | 95 424 | 429 | 434 | 439 | 444 | 448 | 453 | 458 | 463 | 468 | |
| 901 | 472 | 477 | 482 | 487 | 492 | 497 | 501 | 506 | 511 | 516 | |
| 902 | 521 | 525 | 530 | 535 | 540 | 545 | 550 | 554 | 559 | 564 | |
| 903 | 569 | 574 | 578 | 583 | 588 | 593 | 598 | 602 | 607 | 612 | |
| 904 | 617 | 622 | 626 | 631 | 636 | 641 | 646 | 650 | 655 | 660 | |
| 905 | 665 | 670 | 674 | 679 | 684 | 689 | 694 | 698 | 703 | 708 | |
| 906 | 713 | 718 | 722 | 727 | 732 | 737 | 742 | 746 | 751 | 756 | |
| 907 | 761 | 766 | 770 | 775 | 780 | 785 | 789 | 794 | 799 | 804 | |
| 908 | 809 | 813 | 818 | 823 | 828 | 832 | 837 | 842 | 847 | 852 | |
| 909 | 856 | 861 | 866 | 871 | 875 | 880 | 885 | 890 | 895 | 899 | |
| 910 | 904 | 909 | 914 | 918 | 923 | 928 | 933 | 938 | 942 | 947 | |
| 911 | 952 | 957 | 961 | 966 | 971 | 976 | 980 | 985 | 990 | 995 | |
| 912 | 999 | *004 | *009 | *014 | *019 | *023 | *028 | *033 | *038 | *042 | 5 |
| 913 | 96 047 | 052 | 057 | 061 | 066 | 071 | 076 | 080 | 085 | 090 | 1 0.5 |
| 914 | 095 | 099 | 104 | 109 | 114 | 118 | 123 | 128 | 133 | 137 | 2 1.0 |
| 915 | 142 | 147 | 152 | 156 | 161 | 166 | 171 | 175 | 180 | 185 | 3 1.5<br>4 2.0 |
| 916 | 190 | 194 | 199 | 204 | 209 | 213 | 218 | 223 | 227 | 232 | 5 2.5 |
| 917 | 237 | 242 | 246 | 251 | 256 | 261 | 265 | 270 | 275 | 280 | 6 3.0<br>7 3.5 |
| 918 | 284 | 289 | 294 | 298 | 303 | 308 | 313 | 317 | 322 | 327 | 8 4.0 |
| 919 | 332 | 336 | 341 | 346 | 350 | 355 | 360 | 365 | 369 | 374 | 9 4.5 |
| 920 | 379 | 384 | 388 | 393 | 398 | 402 | 407 | 412 | 417 | 421 | |
| 921 | 426 | 431 | 435 | 440 | 445 | 450 | 454 | 459 | 464 | 468 | |
| 922 | 473 | 478 | 483 | 487 | 492 | 497 | 501 | 506 | 511 | 515 | |
| 923 | 520 | 525 | 530 | 534 | 539 | 544 | 548 | 553 | 558 | 562 | |
| 924 | 567 | 572 | 577 | 581 | 586 | 591 | 595 | 600 | 605 | 609 | |
| 925 | 614 | 619 | 624 | 628 | 633 | 638 | 642 | 647 | 652 | 656 | |
| 926 | 661 | 666 | 670 | 675 | 680 | 685 | 689 | 694 | 699 | 703 | |
| 927 | 708 | 713 | 717 | 722 | 727 | 731 | 736 | 741 | 745 | 750 | |
| 928 | 755 | 759 | 764 | 769 | 774 | 778 | 783 | 788 | 792 | 797 | |
| 929 | 802 | 806 | 811 | 816 | 820 | 825 | 830 | 834 | 839 | 844 | |
| 930 | 848 | 853 | 858 | 862 | 867 | 872 | 876 | 881 | 886 | 890 | 4 |
| 931 | 895 | 900 | 904 | 909 | 914 | 918 | 923 | 928 | 932 | 937 | 1 0.4 |
| 932 | 942 | 946 | 951 | 956 | 960 | 965 | 970 | 974 | 979 | 984 | 2 0.8 |
| 933 | 988 | 993 | 997 | *002 | *007 | *011 | *016 | *021 | *025 | *030 | 3 1.2<br>4 1.6 |
| 934 | 97 035 | 039 | 044 | 049 | 053 | 058 | 063 | 067 | 072 | 077 | 5 2.0 |
| 935 | 081 | 086 | 090 | 095 | 100 | 104 | 109 | 114 | 118 | 123 | 6 2.4<br>7 2.8 |
| 936 | 128 | 132 | 137 | 142 | 146 | 151 | 155 | 160 | 165 | 169 | 8 3.2 |
| 937 | 174 | 179 | 183 | 188 | 192 | 197 | 202 | 206 | 211 | 216 | 9 3.6 |
| 938 | 220 | 225 | 230 | 234 | 239 | 243 | 248 | 253 | 257 | 262 | |
| 939 | 267 | 271 | 276 | 280 | 285 | 290 | 294 | 299 | 304 | 308 | |
| 940 | 313 | 317 | 322 | 327 | 331 | 336 | 340 | 345 | 350 | 354 | |
| 941 | 359 | 364 | 368 | 373 | 377 | 382 | 387 | 391 | 396 | 400 | |
| 942 | 405 | 410 | 414 | 419 | 424 | 428 | 433 | 437 | 442 | 447 | |
| 943 | 451 | 456 | 460 | 465 | 470 | 474 | 479 | 483 | 488 | 493 | |
| 944 | 497 | 502 | 506 | 511 | 516 | 520 | 525 | 529 | 534 | 539 | |
| 945 | 543 | 548 | 552 | 557 | 562 | 566 | 571 | 575 | 580 | 585 | |
| 946 | 589 | 594 | 598 | 603 | 607 | 612 | 617 | 621 | 626 | 630 | |
| 947 | 635 | 640 | 644 | 649 | 653 | 658 | 663 | 667 | 672 | 676 | |
| 948 | 681 | 685 | 690 | 695 | 699 | 704 | 708 | 713 | 717 | 722 | |
| 949 | 727 | 731 | 736 | 740 | 745 | 749 | 754 | 759 | 763 | 768 | |

# TABLE II.—LOGARITHMS OF NUMBERS.

| No. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Pp. Pts. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 950 | 97 772 | 777 | 782 | 786 | 791 | 795 | 800 | 804 | 809 | 813 | |
| 951 | 818 | 823 | 827 | 832 | 836 | 841 | 845 | 850 | 855 | 859 | |
| 952 | 864 | 868 | 873 | 877 | 882 | 886 | 891 | 896 | 900 | 905 | |
| 953 | 909 | 914 | 918 | 923 | 928 | 932 | 937 | 941 | 946 | 950 | |
| 954 | 955 | 959 | 964 | 968 | 973 | 978 | 982 | 987 | 991 | 996 | |
| 955 | 98 000 | 005 | 009 | 014 | 019 | 023 | 028 | 032 | 037 | 041 | |
| 956 | 046 | 050 | 055 | 059 | 064 | 068 | 073 | 078 | 082 | 087 | |
| 957 | 091 | 096 | 100 | 105 | 109 | 114 | 118 | 123 | 127 | 132 | |
| 958 | 137 | 141 | 146 | 150 | 155 | 159 | 164 | 168 | 173 | 177 | |
| 959 | 182 | 186 | 191 | 195 | 200 | 204 | 209 | 214 | 218 | 223 | |
| 960 | 227 | 232 | 236 | 241 | 245 | 250 | 254 | 259 | 263 | 268 | |
| 961 | 272 | 277 | 281 | 286 | 290 | 295 | 299 | 304 | 308 | 313 | |
| 962 | 318 | 322 | 327 | 331 | 336 | 340 | 345 | 349 | 354 | 358 | 5 |
| 963 | 363 | 367 | 372 | 376 | 381 | 385 | 390 | 394 | 399 | 403 | 1 0.5 |
| 964 | 408 | 412 | 417 | 421 | 426 | 430 | 435 | 439 | 444 | 448 | 2 1.0 |
| 965 | 453 | 457 | 462 | 466 | 471 | 475 | 480 | 484 | 489 | 493 | 3 1.5 |
| 966 | 498 | 502 | 507 | 511 | 516 | 520 | 525 | 529 | 534 | 538 | 4 2.0 |
| 967 | 543 | 547 | 552 | 556 | 561 | 565 | 570 | 574 | 579 | 583 | 5 2.5 |
| 968 | 588 | 592 | 597 | 601 | 605 | 610 | 614 | 619 | 623 | 628 | 6 3.0 |
| 969 | 632 | 637 | 641 | 646 | 650 | 655 | 659 | 664 | 668 | 673 | 7 3.5 |
| 970 | 677 | 682 | 686 | 691 | 695 | 700 | 704 | 709 | 713 | 717 | 8 4.0 |
| 971 | 722 | 726 | 731 | 735 | 740 | 744 | 749 | 753 | 758 | 762 | 9 4.5 |
| 972 | 767 | 771 | 776 | 780 | 784 | 789 | 793 | 798 | 802 | 807 | |
| 973 | 811 | 816 | 820 | 825 | 829 | 834 | 838 | 843 | 847 | 851 | |
| 974 | 856 | 860 | 865 | 869 | 874 | 878 | 883 | 887 | 892 | 896 | |
| 975 | 900 | 905 | 909 | 914 | 918 | 923 | 927 | 932 | 936 | 941 | |
| 976 | 945 | 949 | 954 | 958 | 963 | 967 | 972 | 976 | 981 | 985 | |
| 977 | 989 | 994 | 998 | *003 | *007 | *012 | *016 | *021 | *025 | *029 | |
| 978 | 99 034 | 038 | 043 | 047 | 052 | 056 | 061 | 065 | 069 | 074 | |
| 979 | 078 | 083 | 087 | 092 | 096 | 100 | 105 | 109 | 114 | 118 | |
| 980 | 123 | 127 | 131 | 136 | 140 | 145 | 149 | 154 | 158 | 162 | |
| 981 | 167 | 171 | 176 | 180 | 185 | 189 | 193 | 198 | 202 | 207 | 4 |
| 982 | 211 | 216 | 220 | 224 | 229 | 233 | 238 | 242 | 247 | 251 | 1 0.4 |
| 983 | 255 | 260 | 264 | 269 | 273 | 277 | 282 | 286 | 291 | 295 | 2 0.8 |
| 984 | 300 | 304 | 308 | 313 | 317 | 322 | 326 | 330 | 335 | 339 | 3 1.2 |
| 985 | 344 | 348 | 352 | 357 | 361 | 366 | 370 | 374 | 379 | 383 | 4 1.6 |
| 986 | 388 | 392 | 396 | 401 | 405 | 410 | 414 | 419 | 423 | 427 | 5 2.0 |
| 987 | 432 | 436 | 441 | 445 | 449 | 454 | 458 | 463 | 467 | 471 | 6 2.4 |
| 988 | 476 | 480 | 484 | 489 | 493 | 498 | 502 | 506 | 511 | 515 | 7 2.8 |
| 989 | 520 | 524 | 528 | 533 | 537 | 542 | 546 | 550 | 555 | 559 | 8 3.2 |
| 990 | 564 | 568 | 572 | 577 | 581 | 585 | 590 | 594 | 599 | 603 | 9 3.6 |
| 991 | 607 | 612 | 616 | 621 | 625 | 629 | 634 | 638 | 642 | 647 | |
| 992 | 651 | 656 | 660 | 664 | 669 | 673 | 677 | 682 | 686 | 691 | |
| 993 | 695 | 699 | 704 | 708 | 712 | 717 | 721 | 726 | 730 | 734 | |
| 994 | 739 | 743 | 747 | 752 | 756 | 760 | 765 | 769 | 774 | 778 | |
| 995 | 782 | 787 | 791 | 795 | 800 | 804 | 808 | 813 | 817 | 822 | |
| 996 | 826 | 830 | 835 | 839 | 843 | 848 | 852 | 856 | 861 | 865 | |
| 997 | 870 | 874 | 878 | 883 | 887 | 891 | 896 | 900 | 904 | 909 | |
| 998 | 913 | 917 | 922 | 926 | 930 | 935 | 939 | 944 | 948 | 952 | |
| 999 | 957 | 961 | 965 | 970 | 974 | 978 | 983 | 987 | 991 | 996 | |

## TABLE III.—BINOMIAL COEFFICIENTS.

### From $n = 1$ to $n = 20$.

$$C_r = \frac{n(n-1)(n-2)\ldots(n-r+1)}{1\times2\times3\times\ldots r}.\quad \text{Also } C_r = C_{n-r} = \frac{n!}{r!\,(n-r)!}.$$

| n | $C_0$ | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ | $C_6$ | $C_7$ | $C_8$ | $C_9$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | | | | | | | | |
| 2 | 1 | 2 | 1 | | | | | | | |
| 3 | 1 | 3 | 3 | 1 | | | | | | |
| 4 | 1 | 4 | 6 | 4 | 1 | | | | | |
| 5 | 1 | 5 | 10 | 10 | 5 | 1 | | | | |
| 6 | 1 | 6 | 15 | 20 | 15 | 6 | 1 | | | |
| 7 | 1 | 7 | 21 | 35 | 35 | 21 | 7 | 1 | | |
| 8 | 1 | 8 | 28 | 56 | 70 | 56 | 28 | 8 | 1 | |
| 9 | 1 | 9 | 36 | 84 | 126 | 126 | 84 | 36 | 9 | 1 |
| 10 | 1 | 10 | 45 | 120 | 210 | 252 | 210 | 120 | 45 | 10 |
| 11 | 1 | 11 | 55 | 165 | 330 | 462 | 462 | 330 | 165 | 55 |
| 12 | 1 | 12 | 66 | 220 | 495 | 792 | 924 | 792 | 495 | 220 |
| 13 | 1 | 13 | 78 | 286 | 715 | 1287 | 1716 | 1716 | 1287 | 715 |
| 14 | 1 | 14 | 91 | 364 | 1001 | 2002 | 3003 | 3432 | 3003 | 2002 |
| 15 | 1 | 15 | 105 | 455 | 1365 | 3003 | 5005 | 6435 | 6435 | 5005 |
| 16 | 1 | 16 | 120 | 560 | 1820 | 4368 | 8008 | 11440 | 12870 | 11440 |
| 17 | 1 | 17 | 136 | 680 | 2380 | 6188 | 12376 | 19448 | 24310 | 24310 |
| 18 | 1 | 18 | 153 | 816 | 3060 | 8568 | 18564 | 31824 | 43758 | 48620 |
| 19 | 1 | 19 | 171 | 969 | 3876 | 11628 | 27132 | 50388 | 75582 | 92378 |
| 20 | 1 | 20 | 190 | 1140 | 4845 | 15504 | 38760 | 77520 | 125970 | 167960 |

| | $C_{10}$ | $C_{11}$ | $C_{12}$ | $C_{13}$ | $C_{14}$ | $C_{15}$ | $C_{16}$ | $C_{17}$ | $C_{18}$ | $C_{19}$ | $C_{20}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 1 | | | | | | | | | | |
| 11 | 11 | 1 | | | | | | | | | |
| 12 | 66 | 12 | 1 | | | | | | | | |
| 13 | 286 | 78 | 13 | 1 | | | | | | | |
| 14 | 1001 | 364 | 91 | 14 | 1 | | | | | | |
| 15 | 3003 | 1365 | 455 | 105 | 15 | 1 | | | | | |
| 16 | 8008 | 4368 | 1820 | 560 | 120 | 16 | 1 | | | | |
| 17 | 19448 | 12376 | 6188 | 2380 | 680 | 136 | 17 | 1 | | | |
| 18 | 43758 | 31824 | 18564 | 8568 | 3060 | 816 | 153 | 18 | 1 | | |
| 19 | 92378 | 75582 | 50388 | 27132 | 11628 | 3876 | 969 | 171 | 19 | 1 | |
| 20 | 184756 | 167960 | 125970 | 77520 | 38760 | 15504 | 4845 | 1140 | 190 | 20 | 1 |

## Factorials.

The continued product of the $n$ factors $1\times2\times3\times4\times\ldots\times n$ is called the factorial of $n$; and it is denoted by the symbol $n!$

Some writers use the symbol $\lfloor n$.

| | | | |
|---|---|---|---|
| $0! =$ | 1 | $11! =$ | 39,916,800 |
| $1! =$ | 1 | $12! =$ | 479,001,600 |
| $2! =$ | 2 | $13! =$ | 6,227,020,800 |
| $3! =$ | 6 | $14! =$ | 87,178,291,200 |
| $4! =$ | 24 | $15! =$ | 1,307,674,368,000 |
| $5! =$ | 120 | $16! =$ | 20,922,789,888,000 |
| $6! =$ | 720 | $17! =$ | 355,687,428,096,000 |
| $7! =$ | 5,040 | $18! =$ | 6,402,373,705,728,000 |
| $8! =$ | 40,320 | $19! =$ | 121,645,100,408,832,000 |
| $9! =$ | 362,880 | $20! =$ | 2,432,902,008,176,640,000 |
| $10! =$ | 3,628,800 | | |

## TABLE IV.—NATURAL LOGARITHMS OF NUMBERS.

| No. | Nat. Log. | No. | Nat. Log. | No. | Nat. Log. | No. | Nat. Log. | No. | Nat. Log. |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.00000 | 51 | 3.93183 | 101 | 4.61512 | 151 | 5.01728 | 201 | 5.30330 |
| 2 | 0.69315 | 52 | 3.95124 | 102 | 4.62497 | 152 | 5.02388 | 202 | 5.30827 |
| 3 | 1.09861 | 53 | 3.97029 | 103 | 4.63473 | 153 | 5.03044 | 203 | 5.31321 |
| 4 | 1.38629 | 54 | 3.98898 | 104 | 4.64439 | 154 | 5.03695 | 204 | 5.31812 |
| 5 | 1.60944 | 55 | 4.00733 | 105 | 4.65396 | 155 | 5.04343 | 205 | 5.32301 |
| 6 | 1.79176 | 56 | 4.02535 | 106 | 4.66344 | 156 | 5.04986 | 206 | 5.32788 |
| 7 | 1.94591 | 57 | 4.04305 | 107 | 4.67283 | 157 | 5.05625 | 207 | 5.33272 |
| 8 | 2.07944 | 58 | 4.06044 | 108 | 4.68213 | 158 | 5.06260 | 208 | 5.33754 |
| 9 | 2.19722 | 59 | 4.07754 | 109 | 4.69135 | 159 | 5.06890 | 209 | 5.34233 |
| 10 | 2.30259 | 60 | 4.09434 | 110 | 4.70048 | 160 | 5.07517 | 210 | 5.34711 |
| 11 | 2.39790 | 61 | 4.11087 | 111 | 4.70953 | 161 | 5.08140 | 211 | 5.35186 |
| 12 | 2.48491 | 62 | 4.12713 | 112 | 4.71850 | 162 | 5.08760 | 212 | 5.35659 |
| 13 | 2.56495 | 63 | 4.14313 | 113 | 4.72739 | 163 | 5.09375 | 213 | 5.36129 |
| 14 | 2.63906 | 64 | 4.15888 | 114 | 4.73620 | 164 | 5.09987 | 214 | 5.36598 |
| 15 | 2.70805 | 65 | 4.17439 | 115 | 4.74493 | 165 | 5.10595 | 215 | 5.37064 |
| 16 | 2.77259 | 66 | 4.18965 | 116 | 4.75359 | 166 | 5.11199 | 216 | 5.37528 |
| 17 | 2.83321 | 67 | 4.20469 | 117 | 4.76217 | 167 | 5.11799 | 217 | 5.37990 |
| 18 | 2.89037 | 68 | 4.21951 | 118 | 4.77068 | 168 | 5.12396 | 218 | 5.38450 |
| 19 | 2.94444 | 69 | 4.23411 | 119 | 4.77912 | 169 | 5.12990 | 219 | 5.38907 |
| 20 | 2.99573 | 70 | 4.24850 | 120 | 4.78749 | 170 | 5.13580 | 220 | 5.39363 |
| 21 | 3.04452 | 71 | 4.26268 | 121 | 4.79579 | 171 | 5.14166 | 221 | 5.39816 |
| 22 | 3.09104 | 72 | 4.27667 | 122 | 4.80402 | 172 | 5.14749 | 222 | 5.40268 |
| 23 | 3.13549 | 73 | 4.29046 | 123 | 4.81218 | 173 | 5.15329 | 223 | 5.40717 |
| 24 | 3.17805 | 74 | 4.30407 | 124 | 4.82028 | 174 | 5.15906 | 224 | 5.41165 |
| 25 | 3.21888 | 75 | 4.31749 | 125 | 4.82831 | 175 | 5.16479 | 225 | 5.41610 |
| 26 | 3.25810 | 76 | 4.33073 | 126 | 4.83628 | 176 | 5.17048 | 226 | 5.42053 |
| 27 | 3.29584 | 77 | 4.34381 | 127 | 4.84419 | 177 | 5.17615 | 227 | 5.42495 |
| 28 | 3.33220 | 78 | 4.35671 | 128 | 4.85203 | 178 | 5.18178 | 228 | 5.42935 |
| 29 | 3.36730 | 79 | 4.36945 | 129 | 4.85981 | 179 | 5.18739 | 229 | 5.43372 |
| 30 | 3.40120 | 80 | 4.38203 | 130 | 4.86753 | 180 | 5.19296 | 230 | 5.43808 |
| 31 | 3.43399 | 81 | 4.39445 | 131 | 4.87520 | 181 | 5.19850 | 231 | 5.44242 |
| 32 | 3.46574 | 82 | 4.40672 | 132 | 4.88280 | 182 | 5.20401 | 232 | 5.44674 |
| 33 | 3.49651 | 83 | 4.41884 | 133 | 4.89035 | 183 | 5.20949 | 233 | 5.45104 |
| 34 | 3.52636 | 84 | 4.43082 | 134 | 4.89784 | 184 | 5.21494 | 234 | 5.45532 |
| 35 | 3.55535 | 85 | 4.44265 | 135 | 4.90527 | 185 | 5.22036 | 235 | 5.45959 |
| 36 | 3.58352 | 86 | 4.45435 | 136 | 4.91265 | 186 | 5.22575 | 236 | 5.46383 |
| 37 | 3.61092 | 87 | 4.46591 | 137 | 4.91998 | 187 | 5.23111 | 237 | 5.46806 |
| 38 | 3.63759 | 88 | 4.47734 | 138 | 4.92725 | 188 | 5.23644 | 238 | 5.47227 |
| 39 | 3.66356 | 89 | 4.48864 | 139 | 4.93447 | 189 | 5.24175 | 239 | 5.47646 |
| 40 | 3.68888 | 90 | 4.49981 | 140 | 4.94164 | 190 | 5.24702 | 240 | 5.48064 |
| 41 | 3.71357 | 91 | 4.51086 | 141 | 4.94876 | 191 | 5.25227 | 241 | 5.48480 |
| 42 | 3.73767 | 92 | 4.52179 | 142 | 4.95583 | 192 | 5.25750 | 242 | 5.48894 |
| 43 | 3.76120 | 93 | 4.53260 | 143 | 4.96284 | 193 | 5.26269 | 243 | 5.49306 |
| 44 | 3.78419 | 94 | 4.54329 | 144 | 4.96981 | 194 | 5.26786 | 244 | 5.49717 |
| 45 | 3.80666 | 95 | 4.55388 | 145 | 4.97673 | 195 | 5.27300 | 245 | 5.50126 |
| 46 | 3.82864 | 96 | 4.56435 | 146 | 4.98361 | 196 | 5.27811 | 246 | 5.50533 |
| 47 | 3.85015 | 97 | 4.57471 | 147 | 4.99043 | 197 | 5.28320 | 247 | 5.50939 |
| 48 | 3.87120 | 98 | 4.58497 | 148 | 4.99721 | 198 | 5.28827 | 248 | 5.51343 |
| 49 | 3.89182 | 99 | 4.59512 | 149 | 5.00395 | 199 | 5.29330 | 249 | 5.51745 |
| 50 | 3.91202 | 100 | 4.60517 | 150 | 5.01064 | 200 | 5.29832 | 250 | 5.52146 |

## TABLE IV.—NATURAL LOGARITHMS OF NUMBERS.

| No. | Nat. Log. | No. | Nat. Log. | No. | Nat. Log. | No. | Nat. Log. | No. | Nat. Log. |
|---|---|---|---|---|---|---|---|---|---|
| 251 | 5.52545 | 301 | 5.70711 | 351 | 5.86079 | 401 | 5.99396 | 451 | 6.11147 |
| 252 | 5.52943 | 302 | 5.71043 | 352 | 5.86363 | 402 | 5.99645 | 452 | 6.11368 |
| 253 | 5.53339 | 303 | 5.71373 | 353 | 5.86647 | 403 | 5.99894 | 453 | 6.11589 |
| 254 | 5.53733 | 304 | 5.71703 | 354 | 5.86930 | 404 | 6.00141 | 454 | 6.11810 |
| 255 | 5.54126 | 305 | 5.72031 | 355 | 5.87212 | 405 | 6.00389 | 455 | 6.12030 |
| 256 | 5.54518 | 306 | 5.72359 | 356 | 5.87493 | 406 | 6.00635 | 456 | 6.12249 |
| 257 | 5.54908 | 307 | 5.72685 | 357 | 5.87774 | 407 | 6.00881 | 457 | 6.12468 |
| 258 | 5.55296 | 308 | 5.73010 | 358 | 5.88053 | 408 | 6.01127 | 458 | 6.12687 |
| 259 | 5.55683 | 309 | 5.73334 | 359 | 5.88332 | 409 | 6.01372 | 459 | 6.12905 |
| 260 | 5.56068 | 310 | 5.73657 | 360 | 5.88610 | 410 | 6.01616 | 460 | 6.13123 |
| 261 | 5.56452 | 311 | 5.73979 | 361 | 5.88888 | 411 | 6.01859 | 461 | 6.13340 |
| 262 | 5.56834 | 312 | 5.74300 | 362 | 5.89164 | 412 | 6.02102 | 462 | 6.13556 |
| 263 | 5.57215 | 313 | 5.74620 | 363 | 5.89440 | 413 | 6.02345 | 463 | 6.13773 |
| 264 | 5.57595 | 314 | 5.74939 | 364 | 5.89715 | 414 | 6.02587 | 464 | 6.13988 |
| 265 | 5.57973 | 315 | 5.75257 | 365 | 5.89990 | 415 | 6.02828 | 465 | 6.14204 |
| 266 | 5.58350 | 316 | 5.75574 | 366 | 5.90263 | 416 | 6.03069 | 466 | 6.14419 |
| 267 | 5.58725 | 317 | 5.75890 | 367 | 5.90536 | 417 | 6.03309 | 467 | 6.14633 |
| 268 | 5.59099 | 318 | 5.76205 | 368 | 5.90808 | 418 | 6.03548 | 468 | 6.14847 |
| 269 | 5.59471 | 319 | 5.76519 | 369 | 5.91080 | 419 | 6.03787 | 469 | 6.15060 |
| 270 | 5.59842 | 320 | 5.76832 | 370 | 5.91350 | 420 | 6.04025 | 470 | 6.15273 |
| 271 | 5.60212 | 321 | 5.77144 | 371 | 5.91620 | 421 | 6.04263 | 471 | 6.15486 |
| 272 | 5.60580 | 322 | 5.77455 | 372 | 5.91889 | 422 | 6.04501 | 472 | 6.15698 |
| 273 | 5.60947 | 323 | 5.77765 | 373 | 5.92158 | 423 | 6.04737 | 473 | 6.15910 |
| 274 | 5.61313 | 324 | 5.78074 | 374 | 5.92426 | 424 | 6.04973 | 474 | 6.16121 |
| 275 | 5.61677 | 325 | 5.78383 | 375 | 5.92693 | 425 | 6.05209 | 475 | 6.16331 |
| 276 | 5.62040 | 326 | 5.78690 | 376 | 5.92959 | 426 | 6.05444 | 476 | 6.16542 |
| 277 | 5.62402 | 327 | 5.78996 | 377 | 5.93225 | 427 | 6.05678 | 477 | 6.16752 |
| 278 | 5.62762 | 328 | 5.79301 | 378 | 5.93489 | 428 | 6.05912 | 478 | 6.16961 |
| 279 | 5.63121 | 329 | 5.79606 | 379 | 5.93754 | 429 | 6.06146 | 479 | 6.17170 |
| 280 | 5.63479 | 330 | 5.79909 | 380 | 5.94017 | 430 | 6.06379 | 480 | 6.17379 |
| 281 | 5.63835 | 331 | 5.80212 | 381 | 5.94280 | 431 | 6.06611 | 481 | 6.17587 |
| 282 | 5.64191 | 332 | 5.80513 | 382 | 5.94542 | 432 | 6.06843 | 482 | 6.17794 |
| 283 | 5.64545 | 333 | 5.80814 | 383 | 5.94803 | 433 | 6.07074 | 483 | 6.18002 |
| 284 | 5.64897 | 334 | 5.81114 | 384 | 5.95064 | 434 | 6.07304 | 484 | 6.18208 |
| 285 | 5.65249 | 335 | 5.81413 | 385 | 5.95324 | 435 | 6.07535 | 485 | 6.18415 |
| 286 | 5.65599 | 336 | 5.81711 | 386 | 5.95584 | 436 | 6.07764 | 486 | 6.18621 |
| 287 | 5.65948 | 337 | 5.82008 | 387 | 5.95842 | 437 | 6.07993 | 487 | 6.18826 |
| 288 | 5.66296 | 338 | 5.82305 | 388 | 5.96101 | 438 | 6.08222 | 488 | 6.19032 |
| 289 | 5.66643 | 339 | 5.82600 | 389 | 5.96358 | 439 | 6.08450 | 489 | 6.19236 |
| 290 | 5.66988 | 340 | 5.82895 | 390 | 5.96615 | 440 | 6.08677 | 490 | 6.19441 |
| 291 | 5.67332 | 341 | 5.83188 | 391 | 5.96871 | 441 | 6.08904 | 491 | 6.19644 |
| 292 | 5.67675 | 342 | 5.83481 | 392 | 5.97126 | 442 | 6.09131 | 492 | 6.19848 |
| 293 | 5.68017 | 343 | 5.83773 | 393 | 5.97381 | 443 | 6.09357 | 493 | 6.20051 |
| 294 | 5.68358 | 344 | 5.84064 | 394 | 5.97635 | 444 | 6.09582 | 494 | 6.20254 |
| 295 | 5.68698 | 345 | 5.84354 | 395 | 5.97889 | 445 | 6.09807 | 495 | 6.20456 |
| 296 | 5.69036 | 346 | 5.84644 | 396 | 5.98141 | 446 | 6.10032 | 496 | 6.20658 |
| 297 | 5.69373 | 347 | 5.84932 | 397 | 5.98394 | 447 | 6.10256 | 497 | 6.20859 |
| 298 | 5.69709 | 348 | 5.85220 | 398 | 5.98645 | 448 | 6.10479 | 498 | 6.21060 |
| 299 | 5.70044 | 349 | 5.85507 | 399 | 5.98896 | 449 | 6.10702 | 499 | 6.21261 |
| 300 | 5.70378 | 350 | 5.85793 | 400 | 5.99146 | 450 | 6.10925 | 500 | 6.21461 |

# TABLE IV.—NATURAL LOGARITHMS OF NUMBERS.

| No. | Nat. Log. | No. | Nat. Log. | No. | Nat. Log. | No. | Nat. Log. | No. | Nat. Log. |
|---|---|---|---|---|---|---|---|---|---|
| 501 | 6.21661 | 551 | 6.31173 | 601 | 6.39859 | 651 | 6.47851 | 701 | 6.55251 |
| 502 | 6.21860 | 552 | 6.31355 | 602 | 6.40026 | 652 | 6.48004 | 702 | 6.55393 |
| 503 | 6.22059 | 553 | 6.31536 | 603 | 6.40192 | 653 | 6.48158 | 703 | 6.55536 |
| 504 | 6.22258 | 554 | 6.31716 | 604 | 6.40357 | 654 | 6.48311 | 704 | 6.55678 |
| 505 | 6.22456 | 555 | 6.31897 | 605 | 6.40523 | 655 | 6.48464 | 705 | 6.55820 |
| 506 | 6.22654 | 556 | 6.32077 | 606 | 6.40688 | 656 | 6.48616 | 706 | 6.55962 |
| 507 | 6.22851 | 557 | 6.32257 | 607 | 6.40853 | 657 | 6.48768 | 707 | 6.56103 |
| 508 | 6.23048 | 558 | 6.32436 | 608 | 6.41017 | 658 | 6.48920 | 708 | 6.56244 |
| 509 | 6.23245 | 559 | 6.32615 | 609 | 6.41182 | 659 | 6.49072 | 709 | 6.56386 |
| 510 | 6.23441 | 560 | 6.32794 | 610 | 6.41346 | 660 | 6.49224 | 710 | 6.56526 |
| 511 | 6.23637 | 561 | 6.32972 | 611 | 6.41510 | 661 | 6.49375 | 711 | 6.56667 |
| 512 | 6.23832 | 562 | 6.33150 | 612 | 6.41673 | 662 | 6.49527 | 712 | 6.56808 |
| 513 | 6.24028 | 563 | 6.33328 | 613 | 6.41836 | 663 | 6.49677 | 713 | 6.56948 |
| 514 | 6.24222 | 564 | 6.33505 | 614 | 6.41999 | 664 | 6.49828 | 714 | 6.57088 |
| 515 | 6.24417 | 565 | 6.33683 | 615 | 6.42162 | 665 | 6.49979 | 715 | 6.57228 |
| 516 | 6.24611 | 566 | 6.33859 | 616 | 6.42325 | 666 | 6.50129 | 716 | 6.57368 |
| 517 | 6.24804 | 567 | 6.34036 | 617 | 6.42487 | 667 | 6.50279 | 717 | 6.57508 |
| 518 | 6.24998 | 568 | 6.34212 | 618 | 6.42649 | 668 | 6.50429 | 718 | 6.57647 |
| 519 | 6.25190 | 569 | 6.34388 | 619 | 6.42811 | 669 | 6.50578 | 719 | 6.57786 |
| 520 | 6.25383 | 570 | 6.34564 | 620 | 6.42972 | 670 | 6.50728 | 720 | 6.57925 |
| 521 | 6.25575 | 571 | 6.34739 | 621 | 6.43133 | 671 | 6.50877 | 721 | 6.58064 |
| 522 | 6.25767 | 572 | 6.34914 | 622 | 6.43294 | 672 | 6.51026 | 722 | 6.58203 |
| 523 | 6.25958 | 573 | 6.35089 | 623 | 6.43455 | 673 | 6.51175 | 723 | 6.58341 |
| 524 | 6.26149 | 574 | 6.35263 | 624 | 6.43615 | 674 | 6.51323 | 724 | 6.58479 |
| 525 | 6.26340 | 575 | 6.35437 | 625 | 6.43775 | 675 | 6.51471 | 725 | 6.58617 |
| 526 | 6.26530 | 576 | 6.35611 | 626 | 6.43935 | 676 | 6.51619 | 726 | 6.58755 |
| 527 | 6.26720 | 577 | 6.35784 | 627 | 6.44095 | 677 | 6.51767 | 727 | 6.58893 |
| 528 | 6.26910 | 578 | 6.35957 | 628 | 6.44254 | 678 | 6.51915 | 728 | 6.59030 |
| 529 | 6.27099 | 579 | 6.36130 | 629 | 6.44413 | 679 | 6.52062 | 729 | 6.59167 |
| 530 | 6.27288 | 580 | 6.36303 | 630 | 6.44572 | 680 | 6.52209 | 730 | 6.59304 |
| 531 | 6.27476 | 581 | 6.36475 | 631 | 6.44731 | 681 | 6.52356 | 731 | 6.59441 |
| 532 | 6.27664 | 582 | 6.36647 | 632 | 6.44889 | 682 | 6.52503 | 732 | 6.59578 |
| 533 | 6.27852 | 583 | 6.36819 | 633 | 6.45047 | 683 | 6.52649 | 733 | 6.59715 |
| 534 | 6.28040 | 584 | 6.36990 | 634 | 6.45205 | 684 | 6.52796 | 734 | 6.59851 |
| 535 | 6.28227 | 585 | 6.37161 | 635 | 6.45362 | 685 | 6.52942 | 735 | 6.59987 |
| 536 | 6.28413 | 586 | 6.37332 | 636 | 6.45520 | 686 | 6.53088 | 736 | 6.60123 |
| 537 | 6.28600 | 587 | 6.37502 | 637 | 6.45677 | 687 | 6.53233 | 737 | 6.60259 |
| 538 | 6.28786 | 588 | 6.37673 | 638 | 6.45834 | 688 | 6.53379 | 738 | 6.60394 |
| 539 | 6.28972 | 589 | 6.37843 | 639 | 6.45990 | 689 | 6.53524 | 739 | 6.60530 |
| 540 | 6.29157 | 590 | 6.38012 | 640 | 6.46147 | 690 | 6.53669 | 740 | 6.60665 |
| 541 | 6.29342 | 591 | 6.38182 | 641 | 6.46303 | 691 | 6.53814 | 741 | 6.60800 |
| 542 | 6.29527 | 592 | 6.38351 | 642 | 6.46459 | 692 | 6.53959 | 742 | 6.60935 |
| 543 | 6.29711 | 593 | 6.38519 | 643 | 6.46614 | 693 | 6.54103 | 743 | 6.61070 |
| 544 | 6.29895 | 594 | 6.38688 | 644 | 6.46770 | 694 | 6.54247 | 744 | 6.61204 |
| 545 | 6.30079 | 595 | 6.38856 | 645 | 6.46925 | 695 | 6.54391 | 745 | 6.61338 |
| 546 | 6.30262 | 596 | 6.39024 | 646 | 6.47080 | 696 | 6.54535 | 746 | 6.61473 |
| 547 | 6.30445 | 597 | 6.39192 | 647 | 6.47235 | 697 | 6.54679 | 747 | 6.61607 |
| 548 | 6.30628 | 598 | 6.39359 | 648 | 6.47389 | 698 | 6.54822 | 748 | 6.61740 |
| 549 | 6.30810 | 599 | 6.39526 | 649 | 6.47543 | 699 | 6.54965 | 749 | 6.61874 |
| 550 | 6.30992 | 600 | 6.39693 | 650 | 6.47697 | 700 | 6.55108 | 750 | 6.62007 |

## TABLE IV. — NATURAL LOGARITHMS OF NUMBERS.

| No. | Nat. Log. | No. | Nat. Log. | No. | Nat. Log. | No. | Nat. Log. | No. | Nat. Log. |
|---|---|---|---|---|---|---|---|---|---|
| 751 | 6.62141 | 801 | 6.68586 | 851 | 6.74641 | 901 | 6.80351 | 951 | 6.85751 |
| 752 | 6.62274 | 802 | 6.68711 | 852 | 6.74759 | 902 | 6.80461 | 952 | 6.85857 |
| 753 | 6.62407 | 803 | 6.68835 | 853 | 6.74876 | 903 | 6.80572 | 953 | 6.85961 |
| 754 | 6.62539 | 804 | 6.68960 | 854 | 6.74993 | 904 | 6.80683 | 954 | 6.86066 |
| 755 | 6.62672 | 805 | 6.69084 | 855 | 6.75110 | 905 | 6.80793 | 955 | 6.86171 |
| 756 | 6.62804 | 806 | 6.69208 | 856 | 6.75227 | 906 | 6.80904 | 956 | 6.86276 |
| 757 | 6.62936 | 807 | 6.69332 | 857 | 6.75344 | 907 | 6.81014 | 957 | 6.86380 |
| 758 | 6.63068 | 808 | 6.69456 | 858 | 6.75460 | 908 | 6.81124 | 958 | 6.86485 |
| 759 | 6.63200 | 809 | 6.69580 | 859 | 6.75577 | 909 | 6.81235 | 959 | 6.86589 |
| 760 | 6.63332 | 810 | 6.69703 | 860 | 6.75693 | 910 | 6.81344 | 960 | 6.86693 |
| 761 | 6.63463 | 811 | 6.69827 | 861 | 6.75809 | 911 | 6.81454 | 961 | 6.86797 |
| 762 | 6.63595 | 812 | 6.69950 | 862 | 6.75926 | 912 | 6.81564 | 962 | 6.86901 |
| 763 | 6.63726 | 813 | 6.70073 | 863 | 6.76041 | 913 | 6.81674 | 963 | 6.87005 |
| 764 | 6.63857 | 814 | 6.70196 | 864 | 6.76157 | 914 | 6.81783 | 964 | 6.87109 |
| 765 | 6.63988 | 815 | 6.70319 | 865 | 6.76273 | 915 | 6.81892 | 965 | 6.87213 |
| 766 | 6.64118 | 816 | 6.70441 | 866 | 6.76388 | 916 | 6.82002 | 966 | 6.87316 |
| 767 | 6.64249 | 817 | 6.70564 | 867 | 6.76504 | 917 | 6.82111 | 967 | 6.87420 |
| 768 | 6.64379 | 818 | 6.70686 | 868 | 6.76619 | 918 | 6.82220 | 968 | 6.87523 |
| 769 | 6.64509 | 819 | 6.70808 | 869 | 6.76734 | 919 | 6.82329 | 969 | 6.87626 |
| 770 | 6.64639 | 820 | 6.70930 | 870 | 6.76849 | 920 | 6.82437 | 970 | 6.87730 |
| 771 | 6.64769 | 821 | 6.71052 | 871 | 6.76964 | 921 | 6.82546 | 971 | 6.87833 |
| 772 | 6.64898 | 822 | 6.71174 | 872 | 6.77079 | 922 | 6.82655 | 972 | 6.87936 |
| 773 | 6.65028 | 823 | 6.71296 | 873 | 6.77194 | 923 | 6.82763 | 973 | 6.88038 |
| 774 | 6.65157 | 824 | 6.71417 | 874 | 6.77308 | 924 | 6.82871 | 974 | 6.88141 |
| 775 | 6.65286 | 825 | 6.71538 | 875 | 6.77422 | 925 | 6.82979 | 975 | 6.88244 |
| 776 | 6.65415 | 826 | 6.71659 | 876 | 6.77537 | 926 | 6.83087 | 976 | 6.88346 |
| 777 | 6.65544 | 827 | 6.71780 | 877 | 6.77651 | 927 | 6.83195 | 977 | 6.88449 |
| 778 | 6.65673 | 828 | 6.71901 | 878 | 6.77765 | 928 | 6.83303 | 978 | 6.88551 |
| 779 | 6.65801 | 829 | 6.72022 | 879 | 6.77878 | 929 | 6.83411 | 979 | 6.88653 |
| 780 | 6.65929 | 830 | 6.72143 | 880 | 6.77992 | 930 | 6.83518 | 980 | 6.88755 |
| 781 | 6.66058 | 831 | 6.72263 | 881 | 6.78106 | 931 | 6.83626 | 981 | 6.88857 |
| 782 | 6.66185 | 832 | 6.72383 | 882 | 6.78219 | 932 | 6.83733 | 982 | 6.88959 |
| 783 | 6.66313 | 833 | 6.72503 | 883 | 6.78333 | 933 | 6.83841 | 983 | 6.89061 |
| 784 | 6.66441 | 834 | 6.72623 | 884 | 6.78446 | 934 | 6.83948 | 984 | 6.89163 |
| 785 | 6.66568 | 835 | 6.72743 | 885 | 6.78559 | 935 | 6.84055 | 985 | 6.89264 |
| 786 | 6.66696 | 836 | 6.72863 | 886 | 6.78672 | 936 | 6.84162 | 986 | 6.89366 |
| 787 | 6.66823 | 837 | 6.72982 | 887 | 6.78784 | 937 | 6.84268 | 987 | 6.89467 |
| 788 | 6.66950 | 838 | 6.73102 | 888 | 6.78897 | 938 | 6.84375 | 988 | 6.89568 |
| 789 | 6.67077 | 839 | 6.73221 | 889 | 6.79010 | 939 | 6.84482 | 989 | 6.89669 |
| 790 | 6.67203 | 840 | 6.73340 | 890 | 6.79122 | 940 | 6.84588 | 990 | 6.89770 |
| 791 | 6.67330 | 841 | 6.73459 | 891 | 6.79234 | 941 | 6.84694 | 991 | 6.89871 |
| 792 | 6.67456 | 842 | 6.73578 | 892 | 6.79347 | 942 | 6.84801 | 992 | 6.89972 |
| 793 | 6.67582 | 843 | 6.73697 | 893 | 6.79459 | 943 | 6.84907 | 993 | 6.90073 |
| 794 | 6.67708 | 844 | 6.73815 | 894 | 6.79571 | 944 | 6.85013 | 994 | 6.90174 |
| 795 | 6.67834 | 845 | 6.73934 | 895 | 6.79682 | 945 | 6.85118 | 995 | 6.90274 |
| 796 | 6.67960 | 846 | 6.74052 | 896 | 6.79794 | 946 | 6.85224 | 996 | 6.90375 |
| 797 | 6.68085 | 847 | 6.74170 | 897 | 6.79906 | 947 | 6.85330 | 997 | 6.90475 |
| 798 | 6.68211 | 848 | 6.74288 | 898 | 6.80017 | 948 | 6.85435 | 998 | 6.90575 |
| 799 | 6.68336 | 849 | 6.74406 | 899 | 6.80128 | 949 | 6.85541 | 999 | 6.90675 |
| 800 | 6.68461 | 850 | 6.74524 | 900 | 6.80239 | 950 | 6.85646 | 1000 | 6.90776 |

## TABLE V.—TRIGONOMETRIC OR CIRCULAR FUNCTIONS.

### Natural Values to Three Places of Decimals.

| Angle | Arc | Sin | Csc | Tan | Ctn | Sec | Cos | | |
|---|---|---|---|---|---|---|---|---|---|
| 0° | 0.000 | 0.000 | infinite | 0.000 | infinite | 1.000 | 1.000 | 1.571 | 90° |
| 1 | 0.017 | 0.017 | 57.30 | 0.017 | 57.29 | 1.000 | 1.000 | 1.553 | 89 |
| 2 | 0.035 | 0.035 | 28.65 | 0.035 | 28.64 | 1.001 | 0.999 | 1.536 | 88 |
| 3 | 0.052 | 0.052 | 19.11 | 0.052 | 19.08 | 1.001 | 0.999 | 1.518 | 87 |
| 4 | 0.070 | 0.070 | 14.34 | 0.070 | 14.30 | 1.002 | 0.998 | 1.501 | 86 |
| 5° | 0.087 | 0.087 | 11.47 | 0.087 | 11.43 | 1.004 | 0.996 | 1.484 | 85° |
| 6 | 0.105 | 0.105 | 9.567 | 0.105 | 9.514 | 1.006 | 0.995 | 1.466 | 84 |
| 7 | 0.122 | 0.122 | 8.206 | 0.123 | 8.144 | 1.008 | 0.993 | 1.449 | 83 |
| 8 | 0.140 | 0.139 | 7.185 | 0.141 | 7.115 | 1.010 | 0.990 | 1.431 | 82 |
| 9 | 0.157 | 0.156 | 6.392 | 0.158 | 6.314 | 1.012 | 0.988 | 1.414 | 81 |
| 10° | 0.175 | 0.174 | 5.759 | 0.176 | 5.671 | 1.015 | 0.985 | 1.396 | 80° |
| 11 | 0.192 | 0.191 | 5.241 | 0.194 | 5.145 | 1.019 | 0.982 | 1.379 | 79 |
| 12 | 0.209 | 0.208 | 4.810 | 0.213 | 4.705 | 1.022 | 0.978 | 1.361 | 78 |
| 13 | 0.227 | 0.225 | 4.445 | 0.231 | 4.331 | 1.026 | 0.974 | 1.344 | 77 |
| 14 | 0.244 | 0.242 | 4.134 | 0.249 | 4.011 | 1.031 | 0.970 | 1.326 | 76 |
| 15° | 0.262 | 0.259 | 3.864 | 0.268 | 3.732 | 1.035 | 0.966 | 1.309 | 75° |
| 16 | 0.279 | 0.276 | 3.628 | 0.287 | 3.487 | 1.040 | 0.961 | 1.292 | 74 |
| 17 | 0.297 | 0.292 | 3.420 | 0.306 | 3.271 | 1.046 | 0.956 | 1.274 | 73 |
| 18 | 0.314 | 0.309 | 3.236 | 0.325 | 3.078 | 1.051 | 0.951 | 1.257 | 72 |
| 19 | 0.332 | 0.326 | 3.072 | 0.344 | 2.904 | 1.058 | 0.946 | 1.239 | 71 |
| 20° | 0.349 | 0.342 | 2.924 | 0.364 | 2.747 | 1.064 | 0.940 | 1.222 | 70° |
| 21 | 0.367 | 0.358 | 2.790 | 0.384 | 2.605 | 1.071 | 0.934 | 1.204 | 69 |
| 22 | 0.384 | 0.375 | 2.669 | 0.404 | 2.475 | 1.079 | 0.927 | 1.187 | 68 |
| 23 | 0.401 | 0.391 | 2.559 | 0.424 | 2.356 | 1.086 | 0.921 | 1.169 | 67 |
| 24 | 0.419 | 0.407 | 2.459 | 0.445 | 2.246 | 1.095 | 0.914 | 1.152 | 66 |
| 25° | 0.436 | 0.423 | 2.366 | 0.466 | 2.145 | 1.103 | 0.906 | 1.134 | 65° |
| 26 | 0.454 | 0.438 | 2.281 | 0.488 | 2.050 | 1.113 | 0.899 | 1.117 | 64 |
| 27 | 0.471 | 0.454 | 2.203 | 0.510 | 1.963 | 1.122 | 0.891 | 1.100 | 63 |
| 28 | 0.489 | 0.469 | 2.130 | 0.532 | 1.881 | 1.133 | 0.883 | 1.082 | 62 |
| 29 | 0.506 | 0.485 | 2.063 | 0.554 | 1.804 | 1.143 | 0.875 | 1.065 | 61 |
| 30° | 0.524 | 0.500 | 2.000 | 0.577 | 1.732 | 1.155 | 0.866 | 1.047 | 60° |
| 31 | 0.541 | 0.515 | 1.942 | 0.601 | 1.664 | 1.167 | 0.857 | 1.030 | 59 |
| 32 | 0.559 | 0.530 | 1.887 | 0.625 | 1.600 | 1.179 | 0.848 | 1.012 | 58 |
| 33 | 0.576 | 0.545 | 1.836 | 0.649 | 1.540 | 1.192 | 0.839 | 0.995 | 57 |
| 34 | 0.593 | 0.559 | 1.788 | 0.675 | 1.483 | 1.206 | 0.829 | 0.977 | 56 |
| 35° | 0.611 | 0.574 | 1.743 | 0.700 | 1.428 | 1.221 | 0.819 | 0.960 | 55° |
| 36 | 0.628 | 0.588 | 1.701 | 0.727 | 1.376 | 1.236 | 0.809 | 0.942 | 54 |
| 37 | 0.646 | 0.602 | 1.662 | 0.754 | 1.327 | 1.252 | 0.799 | 0.925 | 53 |
| 38 | 0.663 | 0.616 | 1.624 | 0.781 | 1.280 | 1.269 | 0.788 | 0.908 | 52 |
| 39 | 0.681 | 0.629 | 1.589 | 0.810 | 1.235 | 1.287 | 0.777 | 0.890 | 51 |
| 40° | 0.698 | 0.643 | 1.556 | 0.839 | 1.192 | 1.305 | 0.766 | 0.873 | 50° |
| 41 | 0.716 | 0.656 | 1.524 | 0.869 | 1.150 | 1.325 | 0.755 | 0.855 | 49 |
| 42 | 0.733 | 0.669 | 1.494 | 0.900 | 1.111 | 1.346 | 0.743 | 0.838 | 48 |
| 43 | 0.750 | 0.682 | 1.466 | 0.933 | 1.072 | 1.367 | 0.731 | 0.820 | 47 |
| 44 | 0.768 | 0.695 | 1.440 | 0.966 | 1.036 | 1.390 | 0.719 | 0.803 | 46 |
| 45° | 0.785 | 0.707 | 1.414 | 1.000 | 1.000 | 1.414 | 0.707 | 0.785 | 45° |
| | | Cos | Sec | Ctn | Tan | Csc | Sin | Arc | Angle |

## TABLE VI.—NATURAL SINES.

| Angle | 0′ | 10′ | 20′ | 30′ | 40′ | 50′ | 60′ | |
|---|---|---|---|---|---|---|---|---|
| 0° | 0.00000 | 0.00291 | 0.00582 | 0.00873 | 0.01164 | 0.01454 | 0.01745 | 89° |
| 1 | 0.01745 | 0.02036 | 0.02327 | 0.02618 | 0.02908 | 0.03199 | 0.03490 | 88 |
| 2 | 0.03490 | 0.03781 | 0.04071 | 0.04362 | 0.04653 | 0.04943 | 0.05234 | 87 |
| 3 | 0.05234 | 0.05524 | 0.05814 | 0.06105 | 0.06395 | 0.06685 | 0.06976 | 86 |
| 4 | 0.06976 | 0.07266 | 0.07556 | 0.07846 | 0.08136 | 0.08426 | 0.08716 | 85 |
| 5 | 0.08716 | 0.09005 | 0.09295 | 0.09585 | 0.09874 | 0.10164 | 0.10453 | 84 |
| 6 | 0.10453 | 0.10742 | 0.11031 | 0.11320 | 0.11609 | 0.11898 | 0.12187 | 83 |
| 7 | 0.12187 | 0.12476 | 0.12764 | 0.13053 | 0.13341 | 0.13629 | 0.13917 | 82 |
| 8 | 0.13917 | 0.14205 | 0.14493 | 0.14781 | 0.15069 | 0.15356 | 0.15643 | 81 |
| 9 | 0.15643 | 0.15931 | 0.16218 | 0.16505 | 0.16792 | 0.17078 | 0.17365 | 80° |
| 10° | 0.17365 | 0.17651 | 0.17937 | 0.18224 | 0.18509 | 0.18795 | 0.19081 | 79 |
| 11 | 0.19081 | 0.19366 | 0.19652 | 0.19937 | 0.20222 | 0.20507 | 0.20791 | 78 |
| 12 | 0.20791 | 0.21076 | 0.21360 | 0.21644 | 0.21928 | 0.22212 | 0.22495 | 77 |
| 13 | 0.22495 | 0.22778 | 0.23062 | 0.23345 | 0.23627 | 0.23910 | 0.24192 | 76 |
| 14 | 0.24192 | 0.24474 | 0.24756 | 0.25038 | 0.25320 | 0.25601 | 0.25882 | 75 |
| 15 | 0.25882 | 0.26163 | 0.26443 | 0.26724 | 0.27004 | 0.27284 | 0.27564 | 74 |
| 16 | 0.27564 | 0.27843 | 0.28123 | 0.28402 | 0.28680 | 0.28959 | 0.29237 | 73 |
| 17 | 0.29237 | 0.29515 | 0.29793 | 0.30071 | 0.30348 | 0.30625 | 0.30902 | 72 |
| 18 | 0.30902 | 0.31178 | 0.31454 | 0.31730 | 0.32006 | 0.32282 | 0.32557 | 71 |
| 19 | 0.32557 | 0.32832 | 0.33106 | 0.33381 | 0.33655 | 0.33929 | 0.34202 | 70° |
| 20° | 0.34202 | 0.34475 | 0.34748 | 0.35021 | 0.35293 | 0.35565 | 0.35837 | 69 |
| 21 | 0.35837 | 0.36108 | 0.36379 | 0.36650 | 0.36921 | 0.37191 | 0.37461 | 68 |
| 22 | 0.37461 | 0.37730 | 0.37999 | 0.38268 | 0.38537 | 0.38805 | 0.39073 | 67 |
| 23 | 0.39073 | 0.39341 | 0.39608 | 0.39875 | 0.40141 | 0.40408 | 0.40674 | 66 |
| 24 | 0.40674 | 0.40939 | 0.41204 | 0.41469 | 0.41734 | 0.41998 | 0.42262 | 65 |
| 25 | 0.42262 | 0.42525 | 0.42788 | 0.43051 | 0.43313 | 0.43575 | 0.43837 | 64 |
| 26 | 0.43837 | 0.44098 | 0.44359 | 0.44620 | 0.44880 | 0.45140 | 0.45399 | 63 |
| 27 | 0.45399 | 0.45658 | 0.45917 | 0.46175 | 0.46433 | 0.46690 | 0.46947 | 62 |
| 28 | 0.46947 | 0.47204 | 0.47460 | 0.47716 | 0.47971 | 0.48226 | 0.48481 | 61 |
| 29 | 0.48481 | 0.48735 | 0.48989 | 0.49242 | 0.49495 | 0.49748 | 0.50000 | 60° |
| 30° | 0.50000 | 0.50252 | 0.50503 | 0.50754 | 0.51004 | 0.51254 | 0.51504 | 59 |
| 31 | 0.51504 | 0.51753 | 0.52002 | 0.52250 | 0.52498 | 0.52745 | 0.52992 | 58 |
| 32 | 0.52992 | 0.53238 | 0.53484 | 0.53730 | 0.53975 | 0.54220 | 0.54464 | 57 |
| 33 | 0.54464 | 0.54708 | 0.54951 | 0.55194 | 0.55436 | 0.55678 | 0.55919 | 56 |
| 34 | 0.55919 | 0.56160 | 0.56401 | 0.56641 | 0.56880 | 0.57119 | 0.57358 | 55 |
| 35 | 0.57358 | 0.57596 | 0.57833 | 0.58070 | 0.58307 | 0.58543 | 0.58779 | 54 |
| 36 | 0.58779 | 0.59014 | 0.59248 | 0.59482 | 0.59716 | 0.59949 | 0.60182 | 53 |
| 37 | 0.60182 | 0.60414 | 0.60645 | 0.60876 | 0.61107 | 0.61337 | 0.61566 | 52 |
| 38 | 0.61566 | 0.61795 | 0.62024 | 0.62251 | 0.62479 | 0.62706 | 0.62932 | 51 |
| 39 | 0.62932 | 0.63158 | 0.63383 | 0.63608 | 0.63832 | 0.64056 | 0.64279 | 50° |
| 40° | 0.64279 | 0.64501 | 0.64723 | 0.64945 | 0.65166 | 0.65386 | 0.65606 | 49 |
| 41 | 0.65606 | 0.65825 | 0.66044 | 0.66262 | 0.66480 | 0.66697 | 0.66913 | 48 |
| 42 | 0.66913 | 0.67129 | 0.67344 | 0.67559 | 0.67773 | 0.67987 | 0.68200 | 47 |
| 43 | 0.68200 | 0.68412 | 0.68624 | 0.68835 | 0.69046 | 0.69256 | 0.69466 | 46 |
| 44 | 0.69466 | 0.69675 | 0.69883 | 0.70091 | 0.70298 | 0.70505 | 0.70711 | 45° |
| | 60′ | 50′ | 40′ | 30′ | 20′ | 10′ | 0′ | Angle |

NATURAL COSINES.

# TABLE VI.—NATURAL SINES.

| Angle | 0′ | 10′ | 20′ | 30′ | 40′ | 50′ | 60′ | |
|---|---|---|---|---|---|---|---|---|
| 45° | 0.70711 | 0.70916 | 0.71121 | 0.71325 | 0.71529 | 0.71732 | 0.71934 | 44 |
| 46 | 0.71934 | 0.72136 | 0.72337 | 0.72537 | 0.72737 | 0.72937 | 0.73135 | 43 |
| 47 | 0.73135 | 0.73333 | 0.73531 | 0.73728 | 0.73924 | 0.74120 | 0.74314 | 42 |
| 48 | 0.74314 | 0.74509 | 0.74703 | 0.74896 | 0.75088 | 0.75280 | 0.75471 | 41 |
| 49 | 0.75471 | 0.75661 | 0.75851 | 0.76041 | 0.76229 | 0.76417 | 0.76604 | 40° |
| 50° | 0.76604 | 0.76791 | 0.76977 | 0.77162 | 0.77347 | 0.77531 | 0.77715 | 39 |
| 51 | 0.77715 | 0.77897 | 0.78079 | 0.78261 | 0.78442 | 0.78622 | 0.78801 | 38 |
| 52 | 0.78801 | 0.78980 | 0.79158 | 0.79335 | 0.79512 | 0.79688 | 0.79864 | 37 |
| 53 | 0.79864 | 0.80038 | 0.80212 | 0.80386 | 0.80558 | 0.80730 | 0.80902 | 36 |
| 54 | 0.80902 | 0.81072 | 0.81242 | 0.81412 | 0.81580 | 0.81748 | 0.81915 | 35 |
| 55 | 0.81915 | 0.82082 | 0.82248 | 0.82413 | 0.82577 | 0.82741 | 0.82904 | 34 |
| 56 | 0.82904 | 0.83066 | 0.83228 | 0.83389 | 0.83549 | 0.83708 | 0.83867 | 33 |
| 57 | 0.83867 | 0.84025 | 0.84182 | 0.84339 | 0.84495 | 0.84650 | 0.84805 | 32 |
| 58 | 0.84805 | 0.84959 | 0.85112 | 0.85264 | 0.85416 | 0.85567 | 0.85717 | 31 |
| 59 | 0.85717 | 0.85866 | 0.86015 | 0.86163 | 0.86310 | 0.86457 | 0.86603 | 30° |
| 60° | 0.86603 | 0.86748 | 0.86892 | 0.87036 | 0.87178 | 0.87321 | 0.87462 | 29 |
| 61 | 0.87462 | 0.87603 | 0.87743 | 0.87882 | 0.88020 | 0.88158 | 0.88295 | 28 |
| 62 | 0.88295 | 0.88431 | 0.88566 | 0.88701 | 0.88835 | 0.88968 | 0.89101 | 27 |
| 63 | 0.89101 | 0.89232 | 0.89363 | 0.89493 | 0.89623 | 0.89752 | 0.89879 | 26 |
| 64 | 0.89879 | 0.90007 | 0.90133 | 0.90259 | 0.90383 | 0.90507 | 0.90631 | 25 |
| 65 | 0.90631 | 0.90753 | 0.90875 | 0.90996 | 0.91116 | 0.91236 | 0.91355 | 24 |
| 66 | 0.91355 | 0.91472 | 0.91590 | 0.91706 | 0.91822 | 0.91936 | 0.92050 | 23 |
| 67 | 0.92050 | 0.92164 | 0.92276 | 0.92388 | 0.92499 | 0.92609 | 0.92718 | 22 |
| 68 | 0.92718 | 0.92827 | 0.92935 | 0.93042 | 0.93148 | 0.93253 | 0.93358 | 21 |
| 69 | 0.93358 | 0.93462 | 0.93565 | 0.93667 | 0.93769 | 0.93869 | 0.93969 | 20° |
| 70° | 0.93969 | 0.94068 | 0.94167 | 0.94264 | 0.94361 | 0.94457 | 0.94552 | 19 |
| 71 | 0.94552 | 0.94646 | 0.94740 | 0.94832 | 0.94924 | 0.95015 | 0.95106 | 18 |
| 72 | 0.95106 | 0.95195 | 0.95284 | 0.95372 | 0.95459 | 0.95545 | 0.95630 | 17 |
| 73 | 0.95630 | 0.95715 | 0.95799 | 0.95882 | 0.95964 | 0.96046 | 0.96126 | 16 |
| 74 | 0.96126 | 0.96206 | 0.96285 | 0.96363 | 0.96440 | 0.96517 | 0.96593 | 15 |
| 75 | 0.96593 | 0.96667 | 0.96742 | 0.96815 | 0.96887 | 0.96959 | 0.97030 | 14 |
| 76 | 0.97030 | 0.97100 | 0.97169 | 0.97237 | 0.97304 | 0.97371 | 0.97437 | 13 |
| 77 | 0.97437 | 0.97502 | 0.97566 | 0.97630 | 0.97692 | 0.97754 | 0.97815 | 12 |
| 78 | 0.97815 | 0.97875 | 0.97934 | 0.97992 | 0.98050 | 0.98107 | 0.98163 | 11 |
| 79 | 0.98163 | 0.98218 | 0.98272 | 0.98325 | 0.98378 | 0.98430 | 0.98481 | 10° |
| 80° | 0.98481 | 0.98531 | 0.98580 | 0.98629 | 0.98676 | 0.98723 | 0.98769 | 9 |
| 81 | 0.98769 | 0.98814 | 0.98858 | 0.98902 | 0.98944 | 0.98986 | 0.99027 | 8 |
| 82 | 0.99027 | 0.99067 | 0.99106 | 0.99144 | 0.99182 | 0.99219 | 0.99255 | 7 |
| 83 | 0.99255 | 0.99290 | 0.99324 | 0.99357 | 0.99390 | 0.99421 | 0.99452 | 6 |
| 84 | 0.99452 | 0.99482 | 0.99511 | 0.99540 | 0.99567 | 0.99594 | 0.99619 | 5 |
| 85 | 0.99619 | 0.99644 | 0.99668 | 0.99692 | 0.99714 | 0.99736 | 0.99756 | 4 |
| 86 | 0.99756 | 0.99776 | 0.99795 | 0.99813 | 0.99831 | 0.99847 | 0.99863 | 3 |
| 87 | 0.99863 | 0.99878 | 0.99892 | 0.99905 | 0.99917 | 0.99929 | 0.99939 | 2 |
| 88 | 0.99939 | 0.99949 | 0.99958 | 0.99966 | 0.99973 | 0.99979 | 0.99985 | 1 |
| 89° | 0.99985 | 0.99989 | 0.99993 | 0.99996 | 0.99998 | 1.00000 | 1.00000 | 0° |
| | 60′ | 50′ | 40′ | 30′ | 20′ | 10′ | 0′ | Angle |

**NATURAL COSINES.**

# TABLE VII. — NATURAL TANGENTS.

| Angle | 0′ | 10′ | 20′ | 30′ | 40′ | 50′ | 60′ | |
|---|---|---|---|---|---|---|---|---|
| 0° | 0.00000 | 0.00291 | 0.00582 | 0.00873 | 0.01164 | 0.01455 | 0.01746 | 89° |
| 1 | 0.01746 | 0.02036 | 0.02328 | 0.02619 | 0.02910 | 0.03201 | 0.03492 | 88 |
| 2 | 0.03492 | 0.03783 | 0.04075 | 0.04366 | 0.04658 | 0.04949 | 0.05241 | 87 |
| 3 | 0.05241 | 0.05533 | 0.05824 | 0.06116 | 0.06408 | 0.06700 | 0.06993 | 86 |
| 4 | 0.06993 | 0.07285 | 0.07578 | 0.07870 | 0.08163 | 0.08456 | 0.08749 | 85 |
| 5 | 0.08749 | 0.09042 | 0.09335 | 0.09629 | 0.09923 | 0.10216 | 0.10510 | 84 |
| 6 | 0.10510 | 0.10805 | 0.11099 | 0.11394 | 0.11688 | 0.11983 | 0.12278 | 83 |
| 7 | 0.12278 | 0.12574 | 0.12869 | 0.13165 | 0.13461 | 0.13758 | 0.14054 | 82 |
| 8 | 0.14054 | 0.14351 | 0.14648 | 0.14945 | 0.15243 | 0.15540 | 0.15838 | 81 |
| 9 | 0.15838 | 0.16137 | 0.16435 | 0.16734 | 0.17033 | 0.17333 | 0.17633 | 80° |
| 10° | 0.17633 | 0.17933 | 0.18233 | 0.18534 | 0.18835 | 0.19136 | 0.19438 | 79 |
| 11 | 0.19438 | 0.19740 | 0.20042 | 0.20345 | 0.20648 | 0.20952 | 0.21256 | 78 |
| 12 | 0.21256 | 0.21560 | 0.21864 | 0.22169 | 0.22475 | 0.22781 | 0.23087 | 77 |
| 13 | 0.23087 | 0.23393 | 0.23700 | 0.24008 | 0.24316 | 0.24624 | 0.24933 | 76 |
| 14 | 0.24933 | 0.25242 | 0.25552 | 0.25862 | 0.26172 | 0.26483 | 0.26795 | 75 |
| 15 | 0.26795 | 0.27107 | 0.27419 | 0.27732 | 0.28046 | 0.28360 | 0.28675 | 74 |
| 16 | 0.28675 | 0.28990 | 0.29305 | 0.29621 | 0.29938 | 0.30255 | 0.30573 | 73 |
| 17 | 0.30573 | 0.30891 | 0.31210 | 0.31530 | 0.31850 | 0.32171 | 0.32492 | 72 |
| 18 | 0.32492 | 0.32814 | 0.33136 | 0.33460 | 0.33783 | 0.34108 | 0.34433 | 71 |
| 19 | 0.34433 | 0.34758 | 0.35085 | 0.35421 | 0.35740 | 0.36068 | 0.36397 | 70° |
| 20° | 0.36397 | 0.36727 | 0.37057 | 0.37388 | 0.37720 | 0.38053 | 0.38386 | 69 |
| 21 | 0.38386 | 0.38721 | 0.39055 | 0.39391 | 0.39727 | 0.40065 | 0.40403 | 68 |
| 22 | 0.40403 | 0.40741 | 0.41081 | 0.41421 | 0.41763 | 0.42105 | 0.42447 | 67 |
| 23 | 0.42447 | 0.42791 | 0.43136 | 0.43481 | 0.43828 | 0.44175 | 0.44523 | 66 |
| 24 | 0.44523 | 0.44872 | 0.45222 | 0.45573 | 0.45924 | 0.46277 | 0.46631 | 65 |
| 25 | 0.46631 | 0.46985 | 0.47341 | 0.47698 | 0.48055 | 0.48414 | 0.48773 | 64 |
| 26 | 0.48773 | 0.49134 | 0.49495 | 0.49858 | 0.50222 | 0.50587 | 0.50953 | 63 |
| 27 | 0.50953 | 0.51320 | 0.51688 | 0.52057 | 0.52427 | 0.52798 | 0.53171 | 62 |
| 28 | 0.53171 | 0.53545 | 0.53920 | 0.54296 | 0.54673 | 0.55051 | 0.55431 | 61 |
| 29 | 0.55431 | 0.55812 | 0.56194 | 0.56577 | 0.56962 | 0.57348 | 0.57735 | 60° |
| 30° | 0.57735 | 0.58124 | 0.58513 | 0.58905 | 0.59297 | 0.59691 | 0.60086 | 59 |
| 31 | 0.60086 | 0.60483 | 0.60881 | 0.61280 | 0.61681 | 0.62083 | 0.62487 | 58 |
| 32 | 0.62487 | 0.62892 | 0.63299 | 0.63707 | 0.64117 | 0.64528 | 0.64941 | 57 |
| 33 | 0.64941 | 0.65355 | 0.65771 | 0.66189 | 0.66608 | 0.67028 | 0.67451 | 56 |
| 34 | 0.67451 | 0.67875 | 0.68301 | 0.68728 | 0.69157 | 0.69588 | 0.70021 | 55 |
| 35 | 0.70021 | 0.70455 | 0.70891 | 0.71329 | 0.71769 | 0.72211 | 0.72654 | 54 |
| 36 | 0.72654 | 0.73100 | 0.73547 | 0.73996 | 0.74447 | 0.74900 | 0.75355 | 53 |
| 37 | 0.75355 | 0.75812 | 0.76272 | 0.76733 | 0.77196 | 0.77661 | 0.78129 | 52 |
| 38 | 0.78129 | 0.78598 | 0.79070 | 0.79544 | 0.80020 | 0.80498 | 0.80978 | 51 |
| 39 | 0.80978 | 0.81461 | 0.81946 | 0.82434 | 0.82923 | 0.83415 | 0.83910 | 50° |
| 40 | 0.83910 | 0.84407 | 0.84906 | 0.85408 | 0.85912 | 0.86419 | 0.86929 | 49 |
| 41 | 0.86929 | 0.87441 | 0.87955 | 0.88473 | 0.88992 | 0.89515 | 0.90040 | 48 |
| 42 | 0.90040 | 0.90569 | 0.91099 | 0.91633 | 0.92170 | 0.92709 | 0.93252 | 47 |
| 43 | 0.93252 | 0.93797 | 0.94345 | 0.94896 | 0.95451 | 0.96008 | 0.96569 | 46 |
| 44° | 0.96569 | 0.97133 | 0.97700 | 0.98270 | 0.98843 | 0.99420 | 1.00000 | 45° |
| | 60′ | 50′ | 40′ | 30′ | 20′ | 10′ | 0′ | Angle |

NATURAL COTANGENTS.

## TABLE VII. — NATURAL TANGENTS.

| Angle | 0′ | 10′ | 20′ | 30′ | 40′ | 50′ | 60′ | |
|---|---|---|---|---|---|---|---|---|
| 45° | 1.00000 | 1.00583 | 1.01170 | 1.01761 | 1.02355 | 1.02952 | 1.03553 | 44° |
| 46 | 1.03553 | 1.04158 | 1.04766 | 1.05378 | 1.05994 | 1.06613 | 1.07237 | 43 |
| 47 | 1.07237 | 1.07864 | 1.08496 | 1.09131 | 1.09770 | 1.10414 | 1.11061 | 42 |
| 48 | 1.11061 | 1.11713 | 1.12369 | 1.13029 | 1.13694 | 1.14363 | 1.15037 | 41 |
| 49 | 1.15037 | 1.15715 | 1.16398 | 1.17085 | 1.17777 | 1.18474 | 1.19175 | 40° |
| 50° | 1.19175 | 1.19882 | 1.20593 | 1.21310 | 1.22031 | 1.22758 | 1.23490 | 39 |
| 51 | 1.23490 | 1.24227 | 1.24969 | 1.25717 | 1.26471 | 1.27230 | 1.27994 | 38 |
| 52 | 1.27994 | 1.28764 | 1.29541 | 1.30323 | 1.31110 | 1.31904 | 1.32704 | 37 |
| 53 | 1.32704 | 1.33511 | 1.34323 | 1.35142 | 1.35968 | 1.36800 | 1.37638 | 36 |
| 54 | 1.37638 | 1.38484 | 1.39336 | 1.40195 | 1.41061 | 1.41934 | 1.42815 | 35 |
| 55 | 1.42815 | 1.43703 | 1.44598 | 1.45501 | 1.46411 | 1.47330 | 1.48256 | 34 |
| 56 | 1.48256 | 1.49190 | 1.50133 | 1.51084 | 1.52043 | 1.53010 | 1.53987 | 33 |
| 57 | 1.53987 | 1.54972 | 1.55966 | 1.56969 | 1.57981 | 1.59002 | 1.60033 | 32 |
| 58 | 1.60033 | 1.61074 | 1.62125 | 1.63185 | 1.64256 | 1.65337 | 1.66428 | 31 |
| 59 | 1.66428 | 1.67530 | 1.68643 | 1.69766 | 1.70901 | 1.72047 | 1.73205 | 30° |
| 60° | 1.73205 | 1.74375 | 1.75556 | 1.76749 | 1.77955 | 1.79174 | 1.80405 | 29 |
| 61 | 1.80405 | 1.81649 | 1.82906 | 1.84177 | 1.85462 | 1.86760 | 1.88073 | 28 |
| 62 | 1.88073 | 1.89400 | 1.90741 | 1.92098 | 1.93470 | 1.94858 | 1.96261 | 27 |
| 63 | 1.96261 | 1.97681 | 1.99116 | 2.00569 | 2.02039 | 2.03526 | 2.05030 | 26 |
| 64 | 2.05030 | 2.06553 | 2.08094 | 2.09654 | 2.11233 | 2.12832 | 2.14451 | 25 |
| 65 | 2.14451 | 2.16090 | 2.17749 | 2.19430 | 2.21132 | 2.22857 | 2.24604 | 24 |
| 66 | 2.24604 | 2.26374 | 2.28167 | 2.29984 | 2.31826 | 2.33693 | 2.35585 | 23 |
| 67 | 2.35585 | 2.37504 | 2.39449 | 2.41421 | 2.43422 | 2.45451 | 2.47509 | 22 |
| 68 | 2.47509 | 2.49597 | 2.51715 | 2.53865 | 2.56046 | 2.58261 | 2.60509 | 21 |
| 69 | 2.60509 | 2.62791 | 2.65109 | 2.67462 | 2.69853 | 2.72281 | 2.74748 | 20° |
| 70° | 2.74748 | 2.77254 | 2.79802 | 2.82391 | 2.85023 | 2.87700 | 2.90421 | 19 |
| 71 | 2.90421 | 2.93189 | 2.96004 | 2.98869 | 3.01783 | 3.04749 | 3.07768 | 18 |
| 72 | 3.07768 | 3.10842 | 3.13972 | 3.17159 | 3.20406 | 3.23714 | 3.27085 | 17 |
| 73 | 3.27085 | 3.30521 | 3.34023 | 3.37594 | 3.41236 | 3.44951 | 3.48741 | 16 |
| 74 | 3.48741 | 3.52609 | 3.56557 | 3.60588 | 3.64705 | 3.68909 | 3.73205 | 15 |
| 75 | 3.73205 | 3.77595 | 3.82083 | 3.86671 | 3.91364 | 3.96165 | 4.01078 | 14 |
| 76 | 4.01078 | 4.06107 | 4.11256 | 4.16530 | 4.21933 | 4.27471 | 4.33148 | 13 |
| 77 | 4.33148 | 4.38969 | 4.44942 | 4.51071 | 4.57363 | 4.63825 | 4.70463 | 12 |
| 78 | 4.70463 | 4.77286 | 4.84300 | 4.91516 | 4.98940 | 5.06584 | 5.14455 | 11 |
| 79 | 5.14455 | 5.22566 | 5.30928 | 5.39552 | 5.48451 | 5.57638 | 5.67128 | 10° |
| 80° | 5.67128 | 5.76937 | 5.87080 | 5.97576 | 6.08444 | 6.19703 | 6.31375 | 9 |
| 81 | 6.31375 | 6.43484 | 6.56055 | 6.69116 | 6.82694 | 6.96823 | 7.11537 | 8 |
| 82 | 7.11537 | 7.26873 | 7.42871 | 7.59575 | 7.77035 | 7.95302 | 8.14435 | 7 |
| 83 | 8.14435 | 8.34496 | 8.55555 | 8.77689 | 9.00983 | 9.25530 | 9.51436 | 6 |
| 84 | 9.51436 | 9.78817 | 10.07803 | 10.38540 | 10.71191 | 11.05943 | 11.43005 | 5 |
| 85 | 11.43005 | 11.82617 | 12.25051 | 12.70621 | 13.19688 | 13.72674 | 14.30067 | 4 |
| 86 | 14.30067 | 14.92442 | 15.60478 | 16.34986 | 17.16934 | 18.07498 | 19.08114 | 3 |
| 87 | 19.08114 | 20.20555 | 21.47040 | 22.90377 | 24.54176 | 26.43160 | 28.63625 | 2 |
| 88 | 28.63625 | 31.24158 | 34.36777 | 38.18846 | 42.96408 | 49.10388 | 57.28996 | 1 |
| 89° | 57.28996 | 68.75009 | 85.93979 | 114.58865 | 171.88540 | 343.77371 | Infinite. | 0° |
| | 60′ | 50′ | 40′ | 30′ | 20′ | 10′ | 0′ | Angle |

NATURAL COTANGENTS.

## TABLE VIII. — NATURAL SECANTS.

| Angle | 0′ | 10′ | 20′ | 30′ | 40′ | 50′ | 60′ | |
|---|---|---|---|---|---|---|---|---|
| 0° | 1.00000 | 1.00001 | 1.00002 | 1.00004 | 1.00007 | 1.00011 | 1.00015 | 89° |
| 1 | 1.00015 | 1.00021 | 1.00027 | 1.00034 | 1.00042 | 1.00051 | 1.00061 | 88 |
| 2 | 1.00061 | 1.00072 | 1.00083 | 1.00095 | 1.00108 | 1.00122 | 1.00137 | 87 |
| 3 | 1.00137 | 1.00153 | 1.00169 | 1.00187 | 1.00205 | 1.00224 | 1.00244 | 86 |
| 4 | 1.00244 | 1.00265 | 1.00287 | 1.00309 | 1.00333 | 1.00357 | 1.00382 | 85 |
| 5 | 1.00382 | 1.00408 | 1.00435 | 1.00463 | 1.00491 | 1.00521 | 1.00551 | 84 |
| 6 | 1.00551 | 1.00582 | 1.00614 | 1.00647 | 1.00681 | 1.00715 | 1.00751 | 83 |
| 7 | 1.00751 | 1.00787 | 1.00825 | 1.00863 | 1.00902 | 1.00942 | 1.00983 | 82 |
| 8 | 1.00983 | 1.01024 | 1.01067 | 1.01111 | 1.01155 | 1.01200 | 1.01247 | 81 |
| 9 | 1.01247 | 1.01294 | 1.01342 | 1.01391 | 1.01440 | 1.01491 | 1.01543 | 80° |
| 10° | 1.01543 | 1.01595 | 1.01649 | 1.01703 | 1.01758 | 1.01815 | 1.01872 | 79 |
| 11 | 1.01872 | 1.01930 | 1.01989 | 1.02049 | 1.02110 | 1.02171 | 1.02234 | 78 |
| 12 | 1.02234 | 1.02298 | 1.02362 | 1.02428 | 1.02494 | 1.02562 | 1.02630 | 77 |
| 13 | 1.02630 | 1.02700 | 1.02770 | 1.02842 | 1.02914 | 1.02987 | 1.03061 | 76 |
| 14 | 1.03061 | 1.03137 | 1.03213 | 1.03290 | 1.03368 | 1.03447 | 1.03528 | 75 |
| 15 | 1.03528 | 1.03609 | 1.03691 | 1.03774 | 1.03858 | 1.03944 | 1.04030 | 74 |
| 16 | 1.04030 | 1.04117 | 1.04206 | 1.04295 | 1.04385 | 1.04477 | 1.04569 | 73 |
| 17 | 1.04569 | 1.04663 | 1.04757 | 1.04853 | 1.04950 | 1.05047 | 1.05146 | 72 |
| 18 | 1.05146 | 1.05246 | 1.05347 | 1.05449 | 1.05552 | 1.05657 | 1.05762 | 71 |
| 19 | 1.05762 | 1.05869 | 1.05976 | 1.06085 | 1.06195 | 1.06306 | 1.06418 | 70° |
| 20° | 1.06418 | 1.06531 | 1.06645 | 1.06761 | 1.06878 | 1.06995 | 1.07115 | 69 |
| 21 | 1.07115 | 1.07235 | 1.07356 | 1.07479 | 1.07602 | 1.07727 | 1.07853 | 68 |
| 22 | 1.07853 | 1.07981 | 1.08109 | 1.08239 | 1.08370 | 1.08503 | 1.08636 | 67 |
| 23 | 1.08636 | 1.08771 | 1.08907 | 1.09044 | 1.09183 | 1.09323 | 1.09464 | 66 |
| 24 | 1.09464 | 1.09606 | 1.09750 | 1.09895 | 1.10041 | 1.10189 | 1.10338 | 65 |
| 25 | 1.10338 | 1.10488 | 1.10640 | 1.10793 | 1.10947 | 1.11103 | 1.11260 | 64 |
| 26 | 1.11260 | 1.11419 | 1.11579 | 1.11740 | 1.11903 | 1.12067 | 1.12233 | 63 |
| 27 | 1.12233 | 1.12400 | 1.12568 | 1.12738 | 1.12910 | 1.13083 | 1.13257 | 62 |
| 28 | 1.13257 | 1.13433 | 1.13610 | 1.13789 | 1.13970 | 1.14152 | 1.14335 | 61 |
| 29 | 1.14335 | 1.14521 | 1.14707 | 1.14896 | 1.15085 | 1.15277 | 1.15470 | 60° |
| 30° | 1.15470 | 1.15665 | 1.15861 | 1.16059 | 1.16259 | 1.16460 | 1.16663 | 59 |
| 31 | 1.16663 | 1.16868 | 1.17075 | 1.17283 | 1.17493 | 1.17704 | 1.17918 | 58 |
| 32 | 1.17918 | 1.18133 | 1.18350 | 1.18569 | 1.18790 | 1.19012 | 1.19236 | 57 |
| 33 | 1.19236 | 1.19463 | 1.19691 | 1.19920 | 1.20152 | 1.20386 | 1.20622 | 56 |
| 34 | 1.20622 | 1.20859 | 1.21099 | 1.21341 | 1.21584 | 1.21830 | 1.22077 | 55 |
| 35 | 1.22077 | 1.22327 | 1.22579 | 1.22833 | 1.23089 | 1.23347 | 1.23607 | 54 |
| 36 | 1.23607 | 1.23869 | 1.24134 | 1.24400 | 1.24669 | 1.24940 | 1.25214 | 53 |
| 37 | 1.25214 | 1.25489 | 1.25767 | 1.26047 | 1.26330 | 1.26615 | 1.26902 | 52 |
| 38 | 1.26902 | 1.27191 | 1.27483 | 1.27778 | 1.28075 | 1.28374 | 1.28676 | 51 |
| 39 | 1.28676 | 1.28980 | 1.29287 | 1.29597 | 1.29909 | 1.30223 | 1.30541 | 50° |
| 40° | 1.30541 | 1.30861 | 1.31183 | 1.31509 | 1.31837 | 1.32168 | 1.32501 | 49 |
| 41 | 1.32501 | 1.32838 | 1.33177 | 1.33519 | 1.33864 | 1.34212 | 1.34563 | 48 |
| 42 | 1.34563 | 1.34917 | 1.35274 | 1.35634 | 1.35997 | 1.36363 | 1.36733 | 47 |
| 43 | 1.36733 | 1.37105 | 1.37481 | 1.37860 | 1.38242 | 1.38628 | 1.39016 | 46 |
| 44° | 1.39016 | 1.39409 | 1.39804 | 1.40203 | 1.40606 | 1.41012 | 1.41421 | 45° |
| | 60′ | 50′ | 40′ | 30′ | 20′ | 10′ | 0′ | Angle |

## NATURAL COSECANTS.

## TABLE VIII. — NATURAL SECANTS.

| Angle | 0′ | 10′ | 20′ | 30′ | 40′ | 50′ | 60′ | |
|---|---|---|---|---|---|---|---|---|
| 45° | 1.41421 | 1.41835 | 1.42251 | 1.42672 | 1.43096 | 1.43524 | 1.43956 | 44° |
| 46 | 1.43956 | 1.44391 | 1.44831 | 1.45274 | 1.45721 | 1.46173 | 1.46628 | 43 |
| 47 | 1.46628 | 1.47087 | 1.47551 | 1.48019 | 1.48491 | 1.48967 | 1.49448 | 42 |
| 48 | 1.49448 | 1.49933 | 1.50422 | 1.50916 | 1.51415 | 1.51918 | 1.52425 | 41 |
| 49 | 1.52425 | 1.52938 | 1.53455 | 1.53977 | 1.54504 | 1.55036 | 1.55572 | 40° |
| 50° | 1.55572 | 1.56114 | 1.56661 | 1.57213 | 1.57771 | 1.58333 | 1.58902 | 39 |
| 51 | 1.58902 | 1.59475 | 1.60054 | 1.60639 | 1.61229 | 1.61825 | 1.62427 | 38 |
| 52 | 1.62427 | 1.63035 | 1.63648 | 1.64268 | 1.64894 | 1.65526 | 1.66164 | 37 |
| 53 | 1.66164 | 1.66809 | 1.67460 | 1.68117 | 1.68782 | 1.69452 | 1.70130 | 36 |
| 54 | 1.70130 | 1.70815 | 1.71506 | 1.72205 | 1.72911 | 1.73624 | 1.74345 | 35 |
| 55 | 1.74345 | 1.75073 | 1.75808 | 1.76552 | 1.77303 | 1.78062 | 1.78829 | 34 |
| 56 | 1.78829 | 1.79604 | 1.80388 | 1.81180 | 1.81981 | 1.82790 | 1.83608 | 33 |
| 57 | 1.83608 | 1.84435 | 1.85271 | 1.86116 | 1.86990 | 1.87834 | 1.88708 | 32 |
| 58 | 1.88708 | 1.89591 | 1.90485 | 1.91388 | 1.92302 | 1.93226 | 1.94160 | 31 |
| 59 | 1.94160 | 1.95106 | 1.96062 | 1.97029 | 1.98008 | 1.98998 | 2.00000 | 30° |
| 60° | 2.00000 | 2.01014 | 2.02039 | 2.03077 | 2.04128 | 2.05191 | 2.06267 | 29 |
| 61 | 2.06267 | 2.07356 | 2.08458 | 2.09574 | 2.10704 | 2.11847 | 2.13005 | 28 |
| 62 | 2.13005 | 2.14178 | 2.15366 | 2.16568 | 2.17786 | 2.19019 | 2.20269 | 27 |
| 63 | 2.20269 | 2.21535 | 2.22817 | 2.24116 | 2.25432 | 2.26766 | 2.28117 | 26 |
| 64 | 2.28117 | 2.29487 | 2.30875 | 2.32282 | 2.33708 | 2.35154 | 2.36620 | 25 |
| 65 | 2.36620 | 2.38107 | 2.39614 | 2.41142 | 2.42692 | 2.44264 | 2.45859 | 24 |
| 66 | 2.45859 | 2.47477 | 2.49119 | 2.50784 | 2.52474 | 2.54190 | 2.55930 | 23 |
| 67 | 2.55930 | 2.57698 | 2.59491 | 2.61313 | 2.63162 | 2.65040 | 2.66947 | 22 |
| 68 | 2.66947 | 2.68884 | 2.70851 | 2.72850 | 2.74881 | 2.76945 | 2.79043 | 21 |
| 69 | 2.79043 | 2.81175 | 2.83342 | 2.85545 | 2.87785 | 2.90063 | 2.92380 | 20° |
| 70° | 2.92380 | 2.94737 | 2.97135 | 2.99574 | 3.02057 | 3.04584 | 3.07155 | 19 |
| 71 | 3.07155 | 3.09774 | 3.12440 | 3.15155 | 3.17920 | 3.20737 | 3.23607 | 18 |
| 72 | 3.23607 | 3.26531 | 3.29512 | 3.32551 | 3.35649 | 3.38808 | 3.42030 | 17 |
| 73 | 3.42030 | 3.45317 | 3.48671 | 3.52094 | 3.55587 | 3.59154 | 3.62796 | 16 |
| 74 | 3.62796 | 3.66515 | 3.70315 | 3.74198 | 3.78166 | 3.82223 | 3.86370 | 15 |
| 75 | 3.86370 | 3.90613 | 3.94952 | 3.99393 | 4.03938 | 4.08591 | 4.13357 | 14 |
| 76 | 4.13357 | 4.18238 | 4.23239 | 4.28366 | 4.33622 | 4.39012 | 4.44541 | 13 |
| 77 | 4.44541 | 4.50216 | 4.56041 | 4.62023 | 4.68167 | 4.74482 | 4.80973 | 12 |
| 78 | 4.80973 | 4.87649 | 4.94517 | 5.01585 | 5.08863 | 5.16359 | 5.24084 | 11 |
| 79 | 5.24084 | 5.32049 | 5.40263 | 5.48740 | 5.57493 | 5.66533 | 5.75877 | 10° |
| 80° | 5.75877 | 5.85539 | 5.95536 | 6.05886 | 6.16607 | 6.27719 | 6.39245 | 9 |
| 81 | 6.39245 | 6.51208 | 6.63633 | 6.76547 | 6.89979 | 7.03962 | 7.18530 | 8 |
| 82 | 7.18530 | 7.33719 | 7.49571 | 7.66130 | 7.83443 | 8.01565 | 8.20551 | 7 |
| 83 | 8.20551 | 8.46466 | 8.61379 | 8.83367 | 9.06515 | 9.30917 | 9.56677 | 6 |
| 84 | 9.56677 | 9.83912 | 10.12752 | 10.43343 | 10.75849 | 11.10455 | 11.47371 | 5 |
| 85 | 11.47371 | 11.86837 | 12.29125 | 12.74550 | 13.23472 | 13.76312 | 14.33559 | 4 |
| 86 | 14.33559 | 14.95788 | 15.63679 | 16.38041 | 17.19843 | 18.10262 | 19.10732 | 3 |
| 87 | 19.10732 | 20.23028 | 21.49368 | 22.92559 | 24.56212 | 26.45051 | 28.65371 | 2 |
| 88 | 28.65371 | 31.25758 | 34.38232 | 38.20155 | 42.97571 | 49.11406 | 57.29869 | 1 |
| 89° | 57.29869 | 68.75736 | 85.94561 | 114.59301 | 171.88831 | 343.77516 | Infinite. | 0° |
| | 60′ | 50′ | 40′ | 30′ | 20′ | 10′ | 0′ | Angle |

## NATURAL COSECANTS.

## TABLE IX. — LOGARITHMS OF TRIGONOMETRIC FUNCTIONS.

| Arc | Angle | Sin | Csc | Tan | Ctn | Sec | Cos | | |
|---|---|---|---|---|---|---|---|---|---|
| 0.0000 | 0°00′ | −∞ | +∞ | −∞ | +∞ | 0.00000 | 0.00000 | 90°00′ | 1.5708 |
| 0.0029 | 10 | 7.46373 | 2.53627 | 7.46373 | 2.53627 | 00000 | 00000 | 50 | 1.5679 |
| 0.0058 | 20 | 76475 | 23525 | 76476 | 23524 | 00001 | 9.99999 | 40 | 1.5650 |
| 0.0087 | 30 | 94084 | 05916 | 94086 | 05914 | 00002 | 99998 | 30 | 1.5621 |
| 0.0116 | 40 | 8.06578 | 1.93422 | 8.06581 | 1.93419 | 00003 | 99997 | 20 | 1.5592 |
| 0.0145 | 50 | 16268 | 83732 | 16273 | 83727 | 00005 | 99995 | 10 | 1.5563 |
| 0.0175 | 1°00′ | 8.24186 | 1.75814 | 8.24192 | 1.75808 | 0.00007 | 9.99993 | 89°00′ | 1.5533 |
| 0.0204 | 10 | 30879 | 69121 | 30888 | 69112 | 00009 | 99991 | 50 | 1.5504 |
| 0.0233 | 20 | 36678 | 63322 | 36689 | 63311 | 00012 | 99988 | 40 | 1.5475 |
| 0.0262 | 30 | 41792 | 58208 | 41807 | 58193 | 00015 | 99985 | 30 | 1.5446 |
| 0.0291 | 40 | 46366 | 53634 | 46385 | 53615 | 00018 | 99982 | 20 | 1.5417 |
| 0.0320 | 50 | 50504 | 49496 | 50527 | 49473 | 00022 | 99978 | 10 | 1.5388 |
| 0.0349 | 2°00′ | 8.54282 | 1.45718 | 8.54308 | 1.45692 | 0.00026 | 9.99974 | 88°00′ | 1.5359 |
| 0.0378 | 10 | 57757 | 42243 | 57788 | 42212 | 00031 | 99969 | 50 | 1.5330 |
| 0.0407 | 20 | 60973 | 39027 | 61009 | 38991 | 00036 | 99964 | 40 | 1.5301 |
| 0.0436 | 30 | 63968 | 36032 | 64009 | 35991 | 00041 | 99959 | 30 | 1.5272 |
| 0.0465 | 40 | 66769 | 33231 | 66816 | 33184 | 00047 | 99953 | 20 | 1.5243 |
| 0.0495 | 50 | 69400 | 30600 | 69453 | 30547 | 00053 | 99947 | 10 | 1.5213 |
| 0.0524 | 3°00′ | 8.71880 | 1.28120 | 8.71940 | 1.28060 | 0.00060 | 9.99940 | 87°00′ | 1.5184 |
| 0.0553 | 10 | 74226 | 25774 | 74292 | 25708 | 00066 | 99934 | 50 | 1.5155 |
| 0.0582 | 20 | 76451 | 23549 | 76525 | 23475 | 00074 | 99926 | 40 | 1.5126 |
| 0.0611 | 30 | 78568 | 21432 | 78649 | 21351 | 00081 | 99919 | 30 | 1.5097 |
| 0.0640 | 40 | 80585 | 19415 | 80674 | 19326 | 00089 | 99911 | 20 | 1.5068 |
| 0.0669 | 50 | 82513 | 17487 | 82610 | 17390 | 00097 | 99903 | 10 | 1.5039 |
| 0.0698 | 4°00′ | 8.84358 | 1.15642 | 8.84464 | 1.15536 | 0.00106 | 9.99894 | 86°00′ | 1.5010 |
| 0.0727 | 10 | 86128 | 13872 | 86243 | 13757 | 00115 | 99885 | 50 | 1.4981 |
| 0.0756 | 20 | 87829 | 12171 | 87953 | 12047 | 00124 | 99876 | 40 | 1.4952 |
| 0.0785 | 30 | 89464 | 10536 | 89598 | 10402 | 00134 | 99866 | 30 | 1.4923 |
| 0.0814 | 40 | 91040 | 08960 | 91185 | 08815 | 00144 | 99856 | 20 | 1.4893 |
| 0.0844 | 50 | 92561 | 07439 | 92716 | 07284 | 00155 | 99845 | 10 | 1.4864 |
| 0.0873 | 5°00′ | 8.94030 | 1.05970 | 8.94195 | 1.05805 | 0.00166 | 9.99834 | 85°00′ | 1.4835 |
| 0.0902 | 10 | 95450 | 04550 | 95627 | 04373 | 00177 | 99823 | 50 | 1.4806 |
| 0.0931 | 20 | 96825 | 03175 | 97013 | 02987 | 00188 | 99812 | 40 | 1.4777 |
| 0.0960 | 30 | 98157 | 01843 | 98358 | 01642 | 00200 | 99800 | 30 | 1.4748 |
| 0.0989 | 40 | 99450 | 00550 | 99662 | 00338 | 00213 | 99787 | 20 | 1.4719 |
| 0.1018 | 50 | 9.00704 | 0.99296 | 9.00930 | 0.99070 | 00225 | 99775 | 10 | 1.4690 |
| 0.1047 | 6°00′ | 9.01923 | 0.98077 | 9.02162 | 0.97838 | 0.00239 | 9.99761 | 84°00′ | 1.4661 |
| 0.1076 | 10 | 03109 | 96891 | 03361 | 96639 | 00252 | 99748 | 50 | 1.4632 |
| 0.1105 | 20 | 04262 | 95738 | 04528 | 95472 | 00266 | 99734 | 40 | 1.4603 |
| 0.1134 | 30 | 05386 | 94614 | 05666 | 94334 | 00280 | 99720 | 30 | 1.4574 |
| 0.1164 | 40 | 06481 | 93519 | 06775 | 93225 | 00295 | 99705 | 20 | 1.4544 |
| 0.1193 | 50 | 07548 | 92452 | 07858 | 92142 | 00310 | 99690 | 10 | 1.4515 |
| 0.1222 | 7°00′ | 9.08589 | 0.91411 | 9.08914 | 0.91086 | 0.00325 | 9.99675 | 83°00′ | 1.4486 |
| 0.1251 | 10 | 09606 | 90394 | 09947 | 90053 | 00341 | 99659 | 50 | 1.4457 |
| 0.1280 | 20 | 10599 | 89401 | 10956 | 89044 | 00357 | 99643 | 40 | 1.4428 |
| 0.1309 | 30 | 11570 | 88430 | 11943 | 88057 | 00373 | 99627 | 30 | 1.4399 |
| 0.1338 | 40 | 12519 | 87481 | 12909 | 87091 | 00390 | 99610 | 20 | 1.4370 |
| 0.1367 | 50 | 13447 | 86553 | 13854 | 86146 | 00407 | 99593 | 10 | 1.4341 |
| 0.1396 | 8°00′ | 9.14356 | 0.85644 | 9.14780 | 0.85220 | 0.00425 | 9.99575 | 82°00′ | 1.4312 |
| 0.1425 | 10 | 15245 | 84755 | 15688 | 84312 | 00443 | 99557 | 50 | 1.4283 |
| 0.1454 | 20 | 16116 | 83884 | 16577 | 83423 | 00461 | 99539 | 40 | 1.4254 |
| 0.1484 | 30 | 16970 | 83030 | 17450 | 82550 | 00480 | 99520 | 30 | 1.4224 |
| 0.1513 | 40 | 17807 | 82193 | 18306 | 81694 | 00499 | 99501 | 20 | 1.4195 |
| 0.1542 | 50 | 18628 | 81372 | 19146 | 80854 | 00518 | 99482 | 10 | 1.4166 |
| 0.1571 | 9°00′ | 9.19433 | 0.80567 | 9.19971 | 0.80029 | 0.00538 | 9.99462 | 81°00′ | 1.4137 |
| | | Cos | Sec | Ctn | Tan | Csc | Sin | Angle | Arc |

## TABLE IX. — LOGARITHMS OF TRIGONOMETRIC FUNCTIONS.

| Arc | Angle | Sin | Csc | Tan | Ctn | Sec | Cos | | |
|---|---|---|---|---|---|---|---|---|---|
| 0.1571 | 9°00′ | 9.19433 | 0.80567 | 9.19971 | 0.80029 | 0.00538 | 9.99462 | 81°00′ | 1.4137 |
| 0.1600 | 10 | 20223 | 79777 | 20782 | 79218 | 00558 | 99442 | 50 | 1.4108 |
| 0.1629 | 20 | 20999 | 79001 | 21578 | 78422 | 00579 | 99421 | 40 | 1.4079 |
| 0.1658 | 30 | 21761 | 78239 | 22361 | 77639 | 00600 | 99400 | 30 | 1.4050 |
| 0.1687 | 40 | 22509 | 77491 | 23130 | 76870 | 00621 | 99379 | 20 | 1.4021 |
| 0.1716 | 50 | 23244 | 76756 | 23887 | 76113 | 00643 | 99357 | 10 | 1.3992 |
| 0.1745 | 10°00′ | 9.23967 | 0.76033 | 9.24632 | 0.75368 | 0.00665 | 9.99335 | 80°00′ | 1.3963 |
| 0.1774 | 10 | 24677 | 75323 | 25365 | 74635 | 00687 | 99313 | 50 | 1.3934 |
| 0.1804 | 20 | 25376 | 74624 | 26086 | 73914 | 00710 | 99290 | 40 | 1.3904 |
| 0.1833 | 30 | 26063 | 73937 | 26797 | 73203 | 00733 | 99267 | 30 | 1.3875 |
| 0.1862 | 40 | 26739 | 73261 | 27496 | 72504 | 00757 | 99243 | 20 | 1.3846 |
| 0.1891 | 50 | 27405 | 72595 | 28186 | 71814 | 00781 | 99219 | 10 | 1.3817 |
| 0.1920 | 11°00′ | 9.28060 | 0.71940 | 9.28865 | 0.71135 | 0.00805 | 9.99195 | 79°00′ | 1.3788 |
| 0.1949 | 10 | 28705 | 71295 | 29535 | 70465 | 00830 | 99170 | 50 | 1.3759 |
| 0.1978 | 20 | 29340 | 70660 | 30195 | 69805 | 00855 | 99145 | 40 | 1.3730 |
| 0.2007 | 30 | 29966 | 70034 | 30846 | 69154 | 00881 | 99119 | 30 | 1.3701 |
| 0.2036 | 40 | 30582 | 69418 | 31489 | 68511 | 00907 | 99093 | 20 | 1.3672 |
| 0.2065 | 50 | 31189 | 68811 | 32122 | 67878 | 00933 | 99067 | 10 | 1.3643 |
| 0.2094 | 12°00′ | 9.31788 | 0.68212 | 9.32747 | 0.67253 | 0.00960 | 9.99040 | 78°00′ | 1.3614 |
| 0.2123 | 10 | 32378 | 67622 | 33365 | 66635 | 00987 | 99013 | 50 | 1.3584 |
| 0.2153 | 20 | 32960 | 67040 | 33974 | 66026 | 01014 | 98986 | 40 | 1.3555 |
| 0.2182 | 30 | 33534 | 66466 | 34576 | 65424 | 01042 | 98958 | 30 | 1.3526 |
| 0.2211 | 40 | 34100 | 65900 | 35170 | 64830 | 01070 | 98930 | 20 | 1.3497 |
| 0.2240 | 50 | 34658 | 65342 | 35757 | 64243 | 01099 | 98901 | 10 | 1.3468 |
| 0.2269 | 13°00′ | 9.35209 | 0.64791 | 9.36336 | 0.63664 | 0.01128 | 9.98872 | 77°00′ | 1.3439 |
| 0.2298 | 10 | 35752 | 64248 | 36909 | 63091 | 01157 | 98843 | 50 | 1.3410 |
| 0.2327 | 20 | 36289 | 63711 | 37476 | 62524 | 01187 | 98813 | 40 | 1.3381 |
| 0.2356 | 30 | 36819 | 63181 | 38035 | 61965 | 01217 | 98783 | 30 | 1.3352 |
| 0.2385 | 40 | 37341 | 62659 | 38589 | 61411 | 01247 | 98753 | 20 | 1.3323 |
| 0.2414 | 50 | 37858 | 62142 | 39136 | 60864 | 01278 | 98722 | 10 | 1.3294 |
| 0.2443 | 14°00′ | 9.38368 | 0.61632 | 9.39677 | 0.60323 | 0.01310 | 9.98690 | 76°00′ | 1.3265 |
| 0.2473 | 10 | 38871 | 61129 | 40212 | 59788 | 01341 | 98659 | 50 | 1.3235 |
| 0.2502 | 20 | 39369 | 60631 | 40742 | 59258 | 01373 | 98627 | 40 | 1.3206 |
| 0.2531 | 30 | 39860 | 60140 | 41266 | 58734 | 01406 | 98594 | 30 | 1.3177 |
| 0.2560 | 40 | 40346 | 59654 | 41784 | 58216 | 01439 | 98561 | 20 | 1.3148 |
| 0.2589 | 50 | 40825 | 59175 | 42297 | 57703 | 01472 | 98528 | 10 | 1.3119 |
| 0.2618 | 15°00′ | 9.41300 | 0.58700 | 9.42805 | 57195 | 0.01506 | 9.98494 | 75°00′ | 1.3090 |
| 0.2647 | 10 | 41768 | 58232 | 43308 | 56692 | 01540 | 98460 | 50 | 1.3061 |
| 0.2676 | 20 | 42232 | 57768 | 43806 | 56194 | 01574 | 98426 | 40 | 1.3032 |
| 0.2705 | 30 | 42690 | 57310 | 44299 | 55701 | 01609 | 98391 | 30 | 1.3003 |
| 0.2734 | 40 | 43143 | 56857 | 44787 | 55213 | 01644 | 98356 | 20 | 1.2974 |
| 0.2763 | 50 | 43591 | 56409 | 45271 | 54729 | 01680 | 98320 | 10 | 1.2945 |
| 0.2793 | 16°00′ | 9.44034 | 0.55966 | 9.45750 | 0.54250 | 0.01716 | 9.98284 | 74°00′ | 1.2915 |
| 0.2822 | 10 | 44472 | 55528 | 46224 | 53776 | 01752 | 98248 | 50 | 1.2886 |
| 0.2851 | 20 | 44905 | 55095 | 46694 | 53306 | 01789 | 98211 | 40 | 1.2857 |
| 0.2880 | 30 | 45334 | 54666 | 47160 | 52840 | 01826 | 98174 | 30 | 1.2828 |
| 0.2909 | 40 | 45758 | 54242 | 47622 | 52378 | 01864 | 98136 | 20 | 1.2799 |
| 0.2938 | 50 | 46178 | 53822 | 48080 | 51920 | 01902 | 98098 | 10 | 1.2770 |
| 0.2967 | 17°00′ | 9.46594 | 0.53406 | 9.48534 | 0.51466 | 0.01940 | 9.98060 | 73°00′ | 1.2741 |
| 0.2996 | 10 | 47005 | 52995 | 48984 | 51016 | 01979 | 98021 | 50 | 1.2712 |
| 0.3025 | 20 | 47411 | 52589 | 49430 | 50570 | 02018 | 97982 | 40 | 1.2683 |
| 0.3054 | 30 | 47814 | 52186 | 49872 | 50128 | 02058 | 97942 | 30 | 1.2654 |
| 0.3083 | 40 | 48213 | 51787 | 50311 | 49689 | 02098 | 97902 | 20 | 1.2625 |
| 0.3113 | 50 | 48607 | 51393 | 50746 | 49254 | 02139 | 97861 | 10 | 1.2595 |
| 0.3142 | 18°00′ | 9.48998 | 0.51002 | 9.51178 | 0.48822 | 0.02179 | 9.97821 | 72°00′ | 1.2566 |
| | | Cos | Sec | Ctn | Tan | Csc | Sin | Angle | Arc |

## TABLE IX.—LOGARITHMS OF TRIGONOMETRIC FUNCTIONS.

| Arc | Angle | Sin | Csc | Tan | Ctn | Sec | Cos | | |
|---|---|---|---|---|---|---|---|---|---|
| 0.3142 | **18°00′** | 9.48998 | 0.51002 | 9.51178 | 0.48822 | 0.02179 | 9.97821 | **72°00′** | 1.2566 |
| 0.3171 | 10 | 49385 | 50615 | 51606 | 48394 | 02221 | 97779 | 50 | 1.2537 |
| 0.3200 | 20 | 49768 | 50232 | 52031 | 47969 | 02262 | 97738 | 40 | 1.2508 |
| 0.3229 | 30 | 50148 | 49852 | 52452 | 47548 | 02304 | 97696 | 30 | 1.2479 |
| 0.3258 | 40 | 50523 | 49477 | 52870 | 47130 | 02347 | 97653 | 20 | 1.2450 |
| 0.3287 | 50 | 50896 | 49104 | 53285 | 46715 | 02390 | 97610 | 10 | 1.2421 |
| 0.3316 | **19°00′** | 9.51264 | 0.48736 | 9.53697 | 0.46303 | 0.02433 | 9.97567 | **71°00′** | 1.2392 |
| 0.3345 | 10 | 51629 | 48371 | 54106 | 45894 | 02477 | 97523 | 50 | 1.2363 |
| 0.3374 | 20 | 51991 | 48009 | 54512 | 45488 | 02521 | 97479 | 40 | 1.2334 |
| 0.3403 | 30 | 52350 | 47650 | 54915 | 45085 | 02565 | 97435 | 30 | 1.2305 |
| 0.3432 | 40 | 52705 | 47295 | 55315 | 44685 | 02610 | 97390 | 20 | 1.2275 |
| 0.3462 | 50 | 53056 | 46944 | 55712 | 44288 | 02656 | 97344 | 10 | 1.2246 |
| 0.3491 | **20°00′** | 9.53405 | 0.46595 | 9.56107 | 0.43893 | 0.02701 | 9.97299 | **70°00′** | 1.2217 |
| 0.3520 | 10 | 53751 | 46249 | 56498 | 43502 | 02748 | 97252 | 50 | 1.2188 |
| 0.3549 | 20 | 54093 | 45907 | 56887 | 43113 | 02794 | 97206 | 40 | 1.2159 |
| 0.3578 | 30 | 54433 | 45567 | 57274 | 42726 | 02841 | 97159 | 30 | 1.2130 |
| 0.3607 | 40 | 54769 | 45231 | 57658 | 42342 | 02889 | 97111 | 20 | 1.2101 |
| 0.3636 | 50 | 55102 | 44898 | 58039 | 41961 | 02937 | 97063 | 10 | 1.2072 |
| 0.3665 | **21°00′** | 9.55433 | 0.44567 | 9.58418 | 0.41582 | 0.02985 | 9.97015 | **69°00′** | 1.2043 |
| 0.3694 | 10 | 55761 | 44239 | 58794 | 41206 | 03034 | 96966 | 50 | 1.2014 |
| 0.3723 | 20 | 56085 | 43915 | 59168 | 40832 | 03083 | 96917 | 40 | 1.1985 |
| 0.3752 | 30 | 56408 | 43592 | 59540 | 40460 | 03132 | 96868 | 30 | 1.1956 |
| 0.3782 | 40 | 56727 | 43273 | 59909 | 40091 | 03182 | 96818 | 20 | 1.1926 |
| 0.3811 | 50 | 57044 | 42956 | 60276 | 39724 | 03233 | 96767 | 10 | 1.1897 |
| 0.3840 | **22°00′** | 9.57358 | 0.42642 | 9.60641 | 0.39359 | 0.03283 | 9.96717 | **68°00′** | 1.1868 |
| 0.3869 | 10 | 57669 | 42331 | 61004 | 38996 | 03335 | 96665 | 50 | 1.1839 |
| 0.3898 | 20 | 57978 | 42022 | 61364 | 38636 | 03386 | 96614 | 40 | 1.1810 |
| 0.3927 | 30 | 58284 | 41716 | 61722 | 38278 | 03438 | 96562 | 30 | 1.1781 |
| 0.3956 | 40 | 58588 | 41412 | 62079 | 37921 | 03491 | 96509 | 20 | 1.1752 |
| 0.3985 | 50 | 58889 | 41111 | 62433 | 37567 | 03544 | 96456 | 10 | 1.1723 |
| 0.4014 | **23°00′** | 9.59188 | 0.40812 | 9.62785 | 0.37215 | 0.03597 | 9.96403 | **67°00′** | 1.1694 |
| 0.4043 | 10 | 59484 | 40516 | 63135 | 36865 | 03651 | 96349 | 50 | 1.1665 |
| 0.4072 | 20 | 59778 | 40222 | 63484 | 36516 | 03706 | 96294 | 40 | 1.1636 |
| 0.4102 | 30 | 60070 | 39930 | 63830 | 36170 | 03760 | 96240 | 30 | 1.1606 |
| 0.4131 | 40 | 60359 | 39641 | 64175 | 35825 | 03815 | 96185 | 20 | 1.1577 |
| 0.4160 | 50 | 60646 | 39354 | 64517 | 35483 | 03871 | 96129 | 10 | 1.1548 |
| 0.4189 | **24°00′** | 9.60931 | 0.39069 | 9.64858 | 0.35142 | 0.03927 | 9.96073 | **66°00′** | 1.1519 |
| 0.4218 | 10 | 61214 | 38786 | 65197 | 34803 | 03983 | 96017 | 50 | 1.1490 |
| 0.4247 | 20 | 61494 | 38506 | 65535 | 34465 | 04040 | 95960 | 40 | 1.1461 |
| 0.4276 | 30 | 61773 | 38227 | 65870 | 34130 | 04098 | 95902 | 30 | 1.1432 |
| 0.4305 | 40 | 62049 | 37951 | 66204 | 33796 | 04156 | 95844 | 20 | 1.1403 |
| 0.4334 | 50 | 62323 | 37677 | 66537 | 33463 | 04214 | 95786 | 10 | 1.1374 |
| 0.4363 | **25°00′** | 9.62595 | 0.37405 | 9.66867 | 0.33133 | 0.04272 | 9.95728 | **65°00′** | 1.1345 |
| 0.4392 | 10 | 62865 | 37135 | 67196 | 32804 | 04332 | 95668 | 50 | 1.1316 |
| 0.4422 | 20 | 63133 | 36867 | 67524 | 32476 | 04391 | 95609 | 40 | 1.1286 |
| 0.4451 | 30 | 63398 | 36602 | 67850 | 32150 | 04451 | 95549 | 30 | 1.1257 |
| 0.4480 | 40 | 63662 | 36338 | 68174 | 31826 | 04512 | 95488 | 20 | 1.1228 |
| 0.4509 | 50 | 63924 | 36076 | 68497 | 31503 | 04573 | 95427 | 10 | 1.1199 |
| 0.4538 | **26°00′** | 9.64184 | 0.35816 | 9.68818 | 0.31182 | 0.04634 | 9.95366 | **64°00′** | 1.1170 |
| 0.4567 | 10 | 64442 | 35558 | 69138 | 30862 | 04696 | 95304 | 50 | 1.1141 |
| 0.4596 | 20 | 64698 | 35302 | 69457 | 30543 | 04758 | 95242 | 40 | 1.1112 |
| 0.4625 | 30 | 64953 | 35047 | 69774 | 30226 | 04821 | 95179 | 30 | 1.1083 |
| 0.4654 | 40 | 65205 | 34795 | 70089 | 29911 | 04884 | 95116 | 20 | 1.1054 |
| 0.4683 | 50 | 65456 | 34544 | 70404 | 29596 | 04948 | 95052 | 10 | 1.1025 |
| 0.4712 | **27°00′** | 9.65705 | 0.34295 | 9.70717 | 0.29283 | 0.05012 | 9.94988 | **63°00′** | 1.0996 |
| | | Cos | Sec | Ctn | Tan | Csc | Sin | Angle | Arc |

## TABLE IX.—LOGARITHMS OF TRIGONOMETRIC FUNCTIONS.

| Arc | Angle | Sin | Csc | Tan | Ctn | Sec | Cos | | |
|---|---|---|---|---|---|---|---|---|---|
| 0.4712 | 27°00′ | 9.65705 | 0.34295 | 9.70717 | 0.29283 | 0.05012 | 9.94988 | 63°00′ | 1.0996 |
| 0.4741 | 10 | 65952 | 34048 | 71028 | 28972 | 05077 | 94923 | 50 | 1.0966 |
| 0.4771 | 20 | 66197 | 33803 | 71339 | 28661 | 05142 | 94858 | 40 | 1.0937 |
| 0.4800 | 30 | 66441 | 33559 | 71648 | 28352 | 05207 | 94793 | 30 | 1.0908 |
| 0.4829 | 40 | 66682 | 33318 | 71955 | 28045 | 05273 | 94727 | 20 | 1.0879 |
| 0.4858 | 50 | 66922 | 33078 | 72262 | 27738 | 05340 | 94660 | 10 | 1.0850 |
| 0.4887 | 28°00′ | 9.67161 | 0.32839 | 9.72567 | 0.27433 | 0.05407 | 9.94593 | 62°00′ | 1.0821 |
| 0.4916 | 10 | 67398 | 32602 | 72872 | 27128 | 05474 | 94526 | 50 | 1.0792 |
| 0.4945 | 20 | 67633 | 32367 | 73175 | 26825 | 05542 | 94458 | 40 | 1.0763 |
| 0.4974 | 30 | 67866 | 32134 | 73476 | 26524 | 05610 | 94390 | 30 | 1.0734 |
| 0.5003 | 40 | 68098 | 31902 | 73777 | 26223 | 05679 | 94321 | 20 | 1.0705 |
| 0.5032 | 50 | 68328 | 31672 | 74077 | 25923 | 05748 | 94252 | 10 | 1.0676 |
| 0.5061 | 29°00′ | 9.68557 | 0.31443 | 9.74375 | 0.25625 | 0.05818 | 9.94182 | 61°00′ | 1.0647 |
| 0.5091 | 10 | 68784 | 31216 | 74673 | 25327 | 05888 | 94112 | 50 | 1.0617 |
| 0.5120 | 20 | 69010 | 30990 | 74969 | 25031 | 05959 | 94041 | 40 | 1.0588 |
| 0.5149 | 30 | 69234 | 30766 | 75264 | 24736 | 06030 | 93970 | 30 | 1.0559 |
| 0.5178 | 40 | 69456 | 30544 | 75558 | 24442 | 06102 | 93898 | 20 | 1.0530 |
| 0.5207 | 50 | 69677 | 30323 | 75852 | 24148 | 06174 | 93826 | 10 | 1.0501 |
| 0.5236 | 30°00′ | 9.69897 | 0.30103 | 9.76144 | 0.23856 | 0.06247 | 9.93753 | 60°00′ | 1.0472 |
| 0.5265 | 10 | 70115 | 29885 | 76435 | 23565 | 06320 | 93680 | 50 | 1.0443 |
| 0.5294 | 20 | 70332 | 29668 | 76725 | 23275 | 06394 | 93606 | 40 | 1.0414 |
| 0.5323 | 30 | 70547 | 29453 | 77015 | 22985 | 06468 | 93532 | 30 | 1.0385 |
| 0.5352 | 40 | 70761 | 29239 | 77303 | 22697 | 06543 | 93457 | 20 | 1.0356 |
| 0.5381 | 50 | 70973 | 29027 | 77591 | 22409 | 06618 | 93382 | 10 | 1.0327 |
| 0.5411 | 31°00′ | 9.71184 | 0.28816 | 9.77877 | 0.22123 | 0.06693 | 9.93307 | 59°00′ | 1.0297 |
| 0.5440 | 10 | 71393 | 28607 | 78163 | 21837 | 06770 | 93230 | 50 | 1.0268 |
| 0.5469 | 20 | 71602 | 28398 | 78448 | 21552 | 06846 | 93154 | 40 | 1.0239 |
| 0.5498 | 30 | 71809 | 28191 | 78732 | 21268 | 06923 | 93077 | 30 | 1.0210 |
| 0.5527 | 40 | 72014 | 27986 | 79015 | 20985 | 07001 | 92999 | 20 | 1.0181 |
| 0.5556 | 50 | 72218 | 27782 | 79297 | 20703 | 07079 | 92921 | 10 | 1.0152 |
| 0.5585 | 32°00′ | 9.72421 | 0.27579 | 9.79579 | 0.20421 | 0.07158 | 9.92842 | 58°00′ | 1.0123 |
| 0.5614 | 10 | 72622 | 27378 | 79860 | 20140 | 07237 | 92763 | 50 | 1.0094 |
| 0.5643 | 20 | 72823 | 27177 | 80140 | 19860 | 07317 | 92683 | 40 | 1.0065 |
| 0.5672 | 30 | 73022 | 26978 | 80419 | 19581 | 07397 | 92603 | 30 | 1.0036 |
| 0.5701 | 40 | 73219 | 26781 | 80697 | 19303 | 07478 | 92522 | 20 | 1.0007 |
| 0.5730 | 50 | 73416 | 26584 | 80975 | 19025 | 07559 | 92441 | 10 | 0.9977 |
| 0.5760 | 33°00′ | 9.73611 | 0.26389 | 9.81252 | 0.18748 | 0.07641 | 9.92359 | 57°00 | 0.9948 |
| 0.5789 | 10 | 73805 | 26195 | 81528 | 18472 | 07723 | 92277 | 50 | 0.9919 |
| 0.5818 | 20 | 73997 | 26003 | 81803 | 18197 | 07806 | 92194 | 40 | 0.9890 |
| 0.5847 | 30 | 74189 | 25811 | 82078 | 17922 | 07889 | 92111 | 30 | 0.9861 |
| 0.5876 | 40 | 74379 | 25621 | 82352 | 17648 | 07973 | 92027 | 20 | 0.9832 |
| 0.5905 | 50 | 74568 | 25432 | 82626 | 17374 | 08058 | 91942 | 10 | 0.9803 |
| 0.5934 | 34°00′ | 9.74756 | 0.25244 | 9.82899 | 0.17101 | 0.08143 | 9.91857 | 56°00′ | 0.9774 |
| 0.5963 | 10 | 74943 | 25057 | 83171 | 16829 | 08228 | 91772 | 50 | 0.9745 |
| 0.5992 | 20 | 75128 | 24872 | 83442 | 16558 | 08314 | 91686 | 40 | 0.9716 |
| 0.6021 | 30 | 75313 | 24687 | 83713 | 16287 | 08401 | 91599 | 30 | 0.9687 |
| 0.6050 | 40 | 75496 | 24504 | 83984 | 16016 | 08488 | 91512 | 20 | 0.9657 |
| 0.6080 | 50 | 75678 | 24322 | 84254 | 15746 | 08575 | 91425 | 10 | 0.9628 |
| 0.6109 | 35°00′ | 9.75859 | 0.24141 | 9.84523 | 0.15477 | 0.08664 | 9.91336 | 55°00′ | 0.9599 |
| 0.6138 | 10 | 76039 | 23961 | 84791 | 15209 | 08752 | 91248 | 50 | 0.9570 |
| 0.6167 | 20 | 76218 | 23782 | 85059 | 14941 | 08842 | 91158 | 40 | 0.9541 |
| 0.6196 | 30 | 76395 | 23605 | 85327 | 14673 | 08931 | 91069 | 30 | 0.9512 |
| 0.6225 | 40 | 76572 | 23428 | 85594 | 14406 | 09022 | 90978 | 20 | 0.9483 |
| 0.6254 | 50 | 76747 | 23253 | 85860 | 14140 | 09113 | 90887 | 10 | 0.9454 |
| 0.6283 | 36°00′ | 9.76922 | 0.23078 | 9.86126 | 0.13874 | 0.09204 | 9.90796 | 54°00′ | 0.9425 |
| | | Cos | Sec | Ctn | Tan | Csc | Sin | Angle | Arc |

## TABLE IX. — LOGARITHMS OF TRIGONOMETRIC FUNCTIONS.

| Arc | Angle | Sin | Csc | Tan | Ctn | Sec | Cos | | |
|---|---|---|---|---|---|---|---|---|---|
| 0.6283 | 36°00′ | 9.76922 | 0.23078 | 9.86126 | 0.13874 | 0.09204 | 9.90796 | 54°00′ | 0.9425 |
| 0.6312 | 10 | 77095 | 22905 | 86392 | 13608 | 09296 | 90704 | 50 | 0.9396 |
| 0.6341 | 20 | 77268 | 22732 | 86656 | 13344 | 09389 | 90611 | 40 | 0.9367 |
| 0.6370 | 30 | 77439 | 22561 | 86921 | 13079 | 09482 | 90518 | 30 | 0.9338 |
| 0.6400 | 40 | 77609 | 22391 | 87185 | 12815 | 09576 | 90424 | 20 | 0.9308 |
| 0.6429 | 50 | 77778 | 22222 | 87448 | 12552 | 09670 | 90330 | 10 | 0.9279 |
| 0.6458 | 37°00′ | 9.77946 | 0.22054 | 9.87711 | 0.12289 | 0.09765 | 9.90235 | 53°00′ | 0.9250 |
| 0.6487 | 10 | 78113 | 21887 | 87974 | 12026 | 09861 | 90139 | 50 | 0.9221 |
| 0.6516 | 20 | 78280 | 21720 | 88236 | 11764 | 09957 | 90043 | 40 | 0.9192 |
| 0.6545 | 30 | 78445 | 21555 | 88498 | 11502 | 10053 | 89947 | 30 | 0.9163 |
| 0.6574 | 40 | 78609 | 21391 | 88759 | 11241 | 10151 | 89849 | 20 | 0.9134 |
| 0.6603 | 50 | 78772 | 21228 | 89020 | 10980 | 10248 | 89752 | 10 | 0.9105 |
| 0.6632 | 38°00′ | 9.78934 | 0.21066 | 9.89281 | 0.10719 | 0.10347 | 9.89653 | 52°00′ | 0.9076 |
| 0.6661 | 10 | 79095 | 20905 | 89541 | 10459 | 10446 | 89554 | 50 | 0.9047 |
| 0.6690 | 20 | 79256 | 20744 | 89801 | 10199 | 10545 | 89455 | 40 | 0.9018 |
| 0.6720 | 30 | 79415 | 20585 | 90061 | 09939 | 10646 | 89354 | 30 | 0.8988 |
| 0.6749 | 40 | 79573 | 20427 | 90320 | 09680 | 10746 | 89254 | 20 | 0.8959 |
| 0.6778 | 50 | 79731 | 20269 | 90578 | 09422 | 10848 | 89152 | 10 | 0.8930 |
| 0.6807 | 39°00′ | 9.79887 | 0.20113 | 9.90837 | 0.09163 | 0.10950 | 9.89050 | 51°00′ | 0.8901 |
| 0.6836 | 10 | 80043 | 19957 | 91095 | 08905 | 11052 | 88948 | 50 | 0.8872 |
| 0.6865 | 20 | 80197 | 19803 | 91353 | 08647 | 11156 | 88844 | 40 | 0.8843 |
| 0.6894 | 30 | 80351 | 19649 | 91610 | 08390 | 11259 | 88741 | 30 | 0.8814 |
| 0.6923 | 40 | 80504 | 19496 | 91868 | 08132 | 11364 | 88636 | 20 | 0.8785 |
| 0.6952 | 50 | 80656 | 19344 | 92125 | 07875 | 11469 | 88531 | 10 | 0.8756 |
| 0.6981 | 40°00′ | 9.80807 | 0.19193 | 9.92381 | 0.07619 | 0.11575 | 9.88425 | 50°00′ | 0.8727 |
| 0.7010 | 10 | 80957 | 19043 | 92638 | 07362 | 11681 | 88319 | 50 | 0.8698 |
| 0.7039 | 20 | 81106 | 18894 | 92894 | 07106 | 11788 | 88212 | 40 | 0.8668 |
| 0.7069 | 30 | 81254 | 18746 | 93150 | 06850 | 11895 | 88105 | 30 | 0.8639 |
| 0.7098 | 40 | 81402 | 18598 | 93406 | 06594 | 12004 | 87996 | 20 | 0.8610 |
| 0.7127 | 50 | 81549 | 18451 | 93661 | 06339 | 12113 | 87887 | 10 | 0.8581 |
| 0.7156 | 41°00′ | 9.81694 | 0.18306 | 9.93916 | 0.06084 | 0.12222 | 9.87778 | 49°00′ | 0.8552 |
| 0.7185 | 10 | 81839 | 18161 | 94171 | 05829 | 12332 | 87668 | 50 | 0.8523 |
| 0.7214 | 20 | 81983 | 18017 | 94426 | 05574 | 12443 | 87557 | 40 | 0.8494 |
| 0.7243 | 30 | 82126 | 17874 | 94681 | 05319 | 12554 | 87446 | 30 | 0.8465 |
| 0.7272 | 40 | 82269 | 17731 | 94935 | 05065 | 12666 | 87334 | 20 | 0.8436 |
| 0.7301 | 50 | 82410 | 17590 | 95190 | 04810 | 12779 | 87221 | 10 | 0.8407 |
| 0.7330 | 42°00′ | 9.82551 | 0.17449 | 9.95444 | 0.04556 | 0.12893 | 9.87107 | 48°00′ | 0.8378 |
| 0.7359 | 10 | 82691 | 17309 | 95698 | 04302 | 13007 | 86993 | 50 | 0.8348 |
| 0.7389 | 20 | 82830 | 17170 | 95952 | 04048 | 13121 | 86879 | 40 | 0.8319 |
| 0.7418 | 30 | 82968 | 17032 | 96205 | 03795 | 13237 | 86763 | 30 | 0.8290 |
| 0.7447 | 40 | 83106 | 16894 | 96459 | 03541 | 13353 | 86647 | 20 | 0.8261 |
| 0.7476 | 50 | 83242 | 16758 | 96712 | 03288 | 13470 | 86530 | 10 | 0.8232 |
| 0.7505 | 43°00′ | 9.83378 | 0.16622 | 9.96966 | 0.03034 | 0.13587 | 9.86413 | 47°00′ | 0.8203 |
| 0.7534 | 10 | 83513 | 16487 | 97219 | 02781 | 13705 | 86295 | 50 | 0.8174 |
| 0.7563 | 20 | 83648 | 16352 | 97472 | 02528 | 13824 | 86176 | 40 | 0.8145 |
| 0.7592 | 30 | 83781 | 16219 | 97725 | 02275 | 13944 | 86056 | 30 | 0.8116 |
| 0.7621 | 40 | 83914 | 16086 | 97978 | 02022 | 14064 | 85936 | 20 | 0.8087 |
| 0.7650 | 50 | 84046 | 15954 | 98231 | 01769 | 14185 | 85815 | 10 | 0.8058 |
| 0.7679 | 44°00′ | 9.84177 | 0.15823 | 9.98484 | 0.01516 | 0.14307 | 9.85693 | 46°00′ | 0.8029 |
| 0.7709 | 10 | 84308 | 15692 | 98737 | 01263 | 14429 | 85571 | 50 | 0.7999 |
| 0.7738 | 20 | 84437 | 15563 | 98989 | 01011 | 14552 | 85448 | 40 | 0.7970 |
| 0.7767 | 30 | 84566 | 15434 | 99242 | 00758 | 14676 | 85324 | 30 | 0.7941 |
| 0.7796 | 40 | 84694 | 15306 | 99495 | 00505 | 14800 | 85200 | 20 | 0.7912 |
| 0.7825 | 50 | 84822 | 15178 | 99747 | 00253 | 14926 | 85074 | 10 | 0.7883 |
| 0.7854 | 45°00′ | 9.84949 | 0.15051 | 0.00000 | 0.00000 | 0.15051 | 9.84949 | 45°00′ | 0.7854 |
| | | Cos | Sec | Ctn | Tan | Csc | Sin | Angle | Arc |

## TABLE X.—ANGLES, ARCS, SINES, TANGENTS, AND SOLID ANGLES.

| Angle $\phi$ in Degrees | Arc, or $\phi$ in Radians | Sine $\phi$ | Tangent $\phi$ | Solid Angle | Angle $\phi$ in Degrees | Arc, or $\phi$ in Radians | Sine $\phi$ | Tangent $\phi$ | Solid Angle |
|---|---|---|---|---|---|---|---|---|---|
| 0° | 0.0000 | 0.0000 | 0.0000 | 0.0000 | 45° | 0.7854 | 0.7071 | 1.0000 | 1.8403 |
| 1 | 0.0175 | 0.0175 | 0.0175 | 0.000957 | 46 | 0.8029 | 0.7193 | 1.0355 | 1.9185 |
| 2 | 0.0349 | 0.0349 | 0.0349 | 0.003828 | 47 | 0.8203 | 0.7314 | 1.0724 | 1.9981 |
| 3 | 0.0524 | 0.0523 | 0.0524 | 0.00861 | 48 | 0.8378 | 0.7431 | 1.1106 | 2.0789 |
| 4 | 0.0698 | 0.0698 | 0.0699 | 0.01531 | 49 | 0.8552 | 0.7547 | 1.1504 | 2.1610 |
| 5° | 0.0873 | 0.0872 | 0.0875 | 0.02391 | 50° | 0.8727 | 0.7660 | 1.1918 | 2.2444 |
| 6 | 0.1047 | 0.1045 | 0.1051 | 0.03442 | 51 | 0.8901 | 0.7771 | 1.2349 | 2.3290 |
| 7 | 0.1222 | 0.1219 | 0.1228 | 0.04683 | 52 | 0.9076 | 0.7880 | 1.2799 | 2.4149 |
| 8 | 0.1396 | 0.1392 | 0.1405 | 0.06115 | 53 | 0.9250 | 0.7986 | 1.3270 | 2.5019 |
| 9 | 0.1571 | 0.1564 | 0.1584 | 0.07736 | 54 | 0.9425 | 0.8090 | 1.3764 | 2.5900 |
| 10° | 0.1745 | 0.1736 | 0.1763 | 0.09546 | 55° | 0.9599 | 0.8192 | 1.4281 | 2.6793 |
| 11 | 0.1920 | 0.1908 | 0.1944 | 0.1154 | 56 | 0.9774 | 0.8290 | 1.4826 | 2.7697 |
| 12 | 0.2094 | 0.2079 | 0.2126 | 0.1373 | 57 | 0.9948 | 0.8387 | 1.5399 | 2.8611 |
| 13 | 0.2269 | 0.2250 | 0.2309 | 0.1610 | 58 | 1.0123 | 0.8480 | 1.6003 | 2.9536 |
| 14 | 0.2443 | 0.2419 | 0.2493 | 0.1866 | 59 | 1.0297 | 0.8572 | 1.6643 | 3.0471 |
| 15° | 0.2618 | 0.2588 | 0.2679 | 0.2141 | 60° | 1.0472 | 0.8660 | 1.7321 | 3.1416 |
| 16 | 0.2793 | 0.2756 | 0.2867 | 0.2434 | 61 | 1.0647 | 0.8746 | 1.8040 | 3.2370 |
| 17 | 0.2967 | 0.2924 | 0.3057 | 0.2745 | 62 | 1.0821 | 0.8829 | 1.8807 | 3.3334 |
| 18 | 0.3142 | 0.3090 | 0.3249 | 0.3075 | 63 | 1.0996 | 0.8910 | 1.9626 | 3.4307 |
| 19 | 0.3316 | 0.3256 | 0.3443 | 0.3423 | 64 | 1.1170 | 0.8988 | 2.0503 | 3.5288 |
| 20° | 0.3491 | 0.3420 | 0.3640 | 0.3789 | 65° | 1.1345 | 0.9063 | 2.1445 | 3.6278 |
| 21 | 0.3665 | 0.3584 | 0.3839 | 0.4173 | 66 | 1.1519 | 0.9135 | 2.2460 | 3.7276 |
| 22 | 0.3840 | 0.3746 | 0.4040 | 0.4575 | 67 | 1.1694 | 0.9205 | 2.3559 | 3.8281 |
| 23 | 0.4014 | 0.3907 | 0.4245 | 0.4995 | 68 | 1.1868 | 0.9272 | 2.4751 | 3.9295 |
| 24 | 0.4189 | 0.4067 | 0.4452 | 0.5432 | 69 | 1.2043 | 0.9336 | 2.6051 | 4.0315 |
| 25° | 0.4363 | 0.4226 | 0.4663 | 0.5887 | 70° | 1.2217 | 0.9397 | 2.7475 | 4.1342 |
| 26 | 0.4538 | 0.4384 | 0.4877 | 0.6359 | 71 | 1.2392 | 0.9455 | 2.9042 | 4.2376 |
| 27 | 0.4712 | 0.4540 | 0.5095 | 0.6848 | 72 | 1.2566 | 0.9511 | 3.0777 | 4.3416 |
| 28 | 0.4887 | 0.4695 | 0.5317 | 0.7355 | 73 | 1.2741 | 0.9563 | 3.2709 | 4.4462 |
| 29 | 0.5061 | 0.4848 | 0.5543 | 0.7878 | 74 | 1.2915 | 0.9613 | 3.4874 | 4.5513 |
| 30° | 0.5236 | 0.5000 | 0.5774 | 0.8418 | 75° | 1.3090 | 0.9659 | 3.7321 | 4.6570 |
| 31 | 0.5411 | 0.5150 | 0.6009 | 0.8974 | 76 | 1.3265 | 0.9703 | 4.0108 | 4.7631 |
| 32 | 0.5585 | 0.5299 | 0.6249 | 0.9547 | 77 | 1.3439 | 0.9744 | 4.3315 | 4.8698 |
| 33 | 0.5760 | 0.5446 | 0.6494 | 1.0137 | 78 | 1.3614 | 0.9781 | 4.7046 | 4.9768 |
| 34 | 0.5934 | 0.5592 | 0.6745 | 1.0742 | 79 | 1.3788 | 0.9816 | 5.1446 | 5.0843 |
| 35° | 0.6109 | 0.5736 | 0.7002 | 1.1363 | 80° | 1.3963 | 0.9848 | 5.6713 | 5.1921 |
| 36 | 0.6283 | 0.5878 | 0.7265 | 1.2000 | 81 | 1.4137 | 0.9877 | 6.3138 | 5.3003 |
| 37 | 0.6458 | 0.6018 | 0.7536 | 1.2652 | 82 | 1.4312 | 0.9903 | 7.1154 | 5.4087 |
| 38 | 0.6632 | 0.6157 | 0.7813 | 1.3320 | 83 | 1.4486 | 0.9925 | 8.1443 | 5.5175 |
| 39 | 0.6807 | 0.6293 | 0.8098 | 1.4002 | 84 | 1.4661 | 0.9945 | 9.5144 | 5.6264 |
| 40° | 0.6981 | 0.6428 | 0.8391 | 1.4700 | 85° | 1.4835 | 0.9962 | 11.4301 | 5.7356 |
| 41 | 0.7156 | 0.6561 | 0.8693 | 1.5412 | 86 | 1.5010 | 0.9976 | 14.3007 | 5.8449 |
| 42 | 0.7330 | 0.6691 | 0.9004 | 1.6139 | 87 | 1.5184 | 0.9986 | 19.0811 | 5.9543 |
| 43 | 0.7505 | 0.6820 | 0.9325 | 1.6880 | 88 | 1.5359 | 0.9994 | 28.6363 | 6.0639 |
| 44 | 0.7679 | 0.6947 | 0.9657 | 1.7634 | 89 | 1.5533 | 0.9998 | 57.2900 | 6.1735 |
| 45° | 0.7854 | 0.7071 | 1.0000 | 1.8403 | 90° | 1.5708 | 1.0000 | Infinite | 6.2832 |

Solid Angle $= 2\pi\ (1 - \cos\phi)$.

*The measure of a solid angle.*—Let a sphere be drawn with its centre at the vertex of any solid angle. The spherical surface intercepted by the faces of the solid angle divided by the square of the radius of the sphere is the measure of the solid angle.

The area of the polar cap extending from the pole down to latitude $90° - \phi$ divided by the square of the radius of the sphere is the measure of the solid angle of the cone whose base is the circle of latitude, and whose vertex is the centre of the sphere.

# TABLE XI.

| Diameter $d$ | Circumference of the Circle. $\pi d$ | Area of the Circle $\frac{1}{4}\pi d^2$ | Volume of the Sphere $\frac{1}{6}\pi d^3$ | Diameter $d$ | Circumference of the Circle $\pi d$ | Area of the Circle $\frac{1}{4}\pi d^2$ | Volume of the Sphere $\frac{1}{6}\pi d^3$ |
|---|---|---|---|---|---|---|---|
| 1 | 3.1416 | 0.7854 | 0.523599 | 51 | 160.22 | 2042.82 | 69455.91 |
| 2 | 6.2832 | 3.1416 | 4.188790 | 52 | 163.36 | 2123.72 | 73622.18 |
| 3 | 9.4248 | 7.0686 | 14.13717 | 53 | 166.50 | 2206.18 | 77951.81 |
| 4 | 12.5664 | 12.5664 | 33.51032 | 54 | 169.65 | 2290.22 | 82447.92 |
| 5 | 15.7080 | 19.6350 | 65.44985 | 55 | 172.79 | 2375.83 | 87113.75 |
| 6 | 18.8496 | 28.2743 | 113.0973 | 56 | 175.93 | 2463.01 | 91952.32 |
| 7 | 21.9911 | 38.4845 | 179.5944 | 57 | 179.07 | 2551.76 | 96966.83 |
| 8 | 25.1327 | 50.2655 | 268.0826 | 58 | 182.21 | 2642.08 | 102160.4 |
| 9 | 28.2743 | 63.6173 | 381.7035 | 59 | 185.35 | 2733.97 | 107536.2 |
| 10 | 31.4159 | 78.5398 | 523.5988 | 60 | 188.50 | 2827.43 | 113097.3 |
| 11 | 34.5575 | 95.0332 | 696.9100 | 61 | 191.64 | 2922.47 | 118847.0 |
| 12 | 37.6991 | 113.097 | 904.7787 | 62 | 194.78 | 3019.07 | 124788.2 |
| 13 | 40.8407 | 132.732 | 1150.347 | 63 | 197.92 | 3117.25 | 130924.3 |
| 14 | 43.9823 | 153.938 | 1436.755 | 64 | 201.06 | 3216.99 | 137258.2 |
| 15 | 47.1239 | 176.715 | 1767.146 | 65 | 204.20 | 3318.31 | 143793.3 |
| 16 | 50.2655 | 201.062 | 2144.660 | 66 | 207.35 | 3421.19 | 150532.6 |
| 17 | 53.4071 | 226.980 | 2572.441 | 67 | 210.49 | 3525.65 | 157479.1 |
| 18 | 56.5487 | 254.469 | 3053.628 | 68 | 213.63 | 3631.68 | 164636.2 |
| 19 | 59.6903 | 283.529 | 3591.364 | 69 | 216.77 | 3739.28 | 172006.9 |
| 20 | 62.8319 | 314.159 | 4188.790 | 70 | 219.91 | 3848.45 | 179594.4 |
| 21 | 65.9734 | 346.361 | 4849.048 | 71 | 223.05 | 3959.19 | 187401.8 |
| 22 | 69.1150 | 380.133 | 5575.280 | 72 | 226.19 | 4071.50 | 195432.2 |
| 23 | 72.2566 | 415.476 | 6370.626 | 73 | 229.34 | 4185.39 | 203688.8 |
| 24 | 75.3982 | 452.389 | 7238.229 | 74 | 232.48 | 4300.84 | 212174.8 |
| 25 | 78.5398 | 490.874 | 8181.231 | 75 | 235.62 | 4417.86 | 220893.2 |
| 26 | 81.6814 | 530.929 | 9202.772 | 76 | 238.76 | 4536.46 | 229847.3 |
| 27 | 84.8230 | 572.555 | 10305.99 | 77 | 241.90 | 4656.63 | 239040.1 |
| 28 | 87.9646 | 615.752 | 11494.04 | 78 | 245.04 | 4778.36 | 248474.9 |
| 29 | 91.1062 | 660.520 | 12770.05 | 79 | 248.19 | 4901.67 | 258154.6 |
| 30 | 94.2478 | 706.858 | 14137.17 | 80 | 251.33 | 5026.55 | 268082.6 |
| 31 | 97.389 | 754.768 | 15598.53 | 81 | 254.47 | 5153.00 | 278261.8 |
| 32 | 100.531 | 804.248 | 17157.28 | 82 | 257.61 | 5281.02 | 288695.6 |
| 33 | 103.673 | 855.299 | 18816.57 | 83 | 260.75 | 5410.61 | 299387.0 |
| 34 | 106.814 | 907.920 | 20579.53 | 84 | 263.89 | 5541.77 | 310339.1 |
| 35 | 109.956 | 962.113 | 22449.30 | 85 | 267.04 | 5674.50 | 321555.1 |
| 36 | 113.097 | 1017.88 | 24429.02 | 86 | 270.18 | 5808.80 | 333038.2 |
| 37 | 116.239 | 1075.21 | 26521.85 | 87 | 273.32 | 5944.68 | 344791.4 |
| 38 | 119.381 | 1134.11 | 28730.91 | 88 | 276.46 | 6082.12 | 356817.9 |
| 39 | 122.522 | 1194.59 | 31059.36 | 89 | 279.60 | 6221.14 | 369120.9 |
| 40 | 125.664 | 1256.64 | 33510.32 | 90 | 282.74 | 6361.73 | 381703.5 |
| 41 | 128.81 | 1320.25 | 36086.95 | 91 | 285.88 | 6503.88 | 394568.9 |
| 42 | 131.95 | 1385.44 | 38792.39 | 92 | 289.03 | 6647.61 | 407720.1 |
| 43 | 135.09 | 1452.20 | 41629.77 | 93 | 292.17 | 6792.91 | 421160.3 |
| 44 | 138.23 | 1520.53 | 44602.24 | 94 | 295.31 | 6939.78 | 434892.8 |
| 45 | 141.37 | 1590.43 | 47712.94 | 95 | 298.45 | 7088.22 | 448920.5 |
| 46 | 144.51 | 1661.90 | 50965.01 | 96 | 301.59 | 7238.23 | 463246.7 |
| 47 | 147.65 | 1734.94 | 54361.60 | 97 | 304.73 | 7389.81 | 477874.5 |
| 48 | 150.80 | 1809.56 | 57905.84 | 98 | 307.88 | 7542.96 | 492807.0 |
| 49 | 153.94 | 1885.74 | 61600.87 | 99 | 311.02 | 7697.69 | 508047.4 |
| 50 | 157.08 | 1963.50 | 65449.85 | 100 | 314.16 | 7853.98 | 523598.8 |

TABLE XI.

| Diameter $d$ | Circumference of the Circle $\pi d$ | Area of the Circle $\frac{1}{4}\pi d^2$ | Volume of the Sphere $\frac{1}{6}\pi d^3$ | Diameter $d$ | Circumference of the Circle $\pi d$ | Area of the Circle $\frac{1}{4}\pi d^2$ | Volume of the Sphere $\frac{1}{6}\pi d^3$ |
|---|---|---|---|---|---|---|---|
| **101** | 317.30 | 8011.85 | 539464.3 | **151** | 474.38 | 17907.9 | 1802725 |
| **102** | 320.44 | 8171.28 | 555647.2 | **152** | 477.52 | 18145.8 | 1838778 |
| **103** | 323.58 | 8332.29 | 572150.5 | **153** | 480.66 | 18385.4 | 1875309 |
| **104** | 326.73 | 8494.87 | 588977.4 | **154** | 483.81 | 18626.5 | 1912321 |
| **105** | 329.87 | 8659.01 | 606131.0 | **155** | 486.95 | 18869.2 | 1949816 |
| **106** | 333.01 | 8824.73 | 623614.5 | **156** | 490.09 | 19113.4 | 1987799 |
| **107** | 336.15 | 8992.02 | 641431.0 | **157** | 493.23 | 19359.3 | 2026271 |
| **108** | 339.29 | 9160.88 | 659583.7 | **158** | 496.37 | 19606.7 | 2065237 |
| **109** | 342.43 | 9331.32 | 678075.6 | **159** | 499.51 | 19855.7 | 2104699 |
| **110** | 345.58 | 9503.32 | 696910.0 | **160** | 502.65 | 20106.2 | 2144660 |
| **111** | 348.72 | 9676.89 | 716090.0 | **161** | 505.80 | 20358.3 | 2185125 |
| **112** | 351.86 | 9852.03 | 735618.6 | **162** | 508.94 | 20612.0 | 2226094 |
| **113** | 355.00 | 10028.7 | 755499.1 | **163** | 512.08 | 20867.2 | 2267574 |
| **114** | 358.14 | 10207.0 | 775734.6 | **164** | 515.22 | 21124.1 | 2309565 |
| **115** | 361.28 | 10386.9 | 796328.3 | **165** | 518.36 | 21382.5 | 2352071 |
| **116** | 364.42 | 10568.3 | 817283.2 | **166** | 521.50 | 21642.4 | 2395096 |
| **117** | 367.57 | 10751.3 | 838602.7 | **167** | 524.65 | 21904.0 | 2438642 |
| **118** | 370.71 | 10935.9 | 860289.5 | **168** | 527.79 | 22167.1 | 2482713 |
| **119** | 373.85 | 11122.0 | 882347.3 | **169** | 530.93 | 22431.8 | 2527311 |
| **120** | 376.99 | 11309.7 | 904778.7 | **170** | 534.07 | 22698.0 | 2572441 |
| **121** | 380.13 | 11499.0 | 927587.2 | **171** | 537.21 | 22965.8 | 2618104 |
| **122** | 383.27 | 11689.9 | 950775.8 | **172** | 540.35 | 23235.2 | 2664305 |
| **123** | 386.42 | 11882.3 | 974347.7 | **173** | 543.50 | 23506.2 | 2711046 |
| **124** | 389.56 | 12076.3 | 998305.9 | **174** | 546.64 | 23778.7 | 2758331 |
| **125** | 392.70 | 12271.8 | 1022654 | **175** | 549.78 | 24052.8 | 2806162 |
| **126** | 395.84 | 12469.0 | 1047394 | **176** | 552.92 | 24328.5 | 2854543 |
| **127** | 398.98 | 12667.7 | 1072531 | **177** | 556.06 | 24605.7 | 2903477 |
| **128** | 402.12 | 12868.0 | 1098066 | **178** | 559.20 | 24884.6 | 2952967 |
| **129** | 405.27 | 13069.8 | 1124004 | **179** | 562.35 | 25164.9 | 3003006 |
| **130** | 408.41 | 13273.2 | 1150347 | **180** | 565.49 | 25446.9 | 3053628 |
| **131** | 411.55 | 13478.2 | 1177098 | **181** | 568.63 | 25730.4 | 3104805 |
| **132** | 414.69 | 13684.8 | 1204260 | **182** | 571.77 | 26015.5 | 3156551 |
| **133** | 417.83 | 13892.9 | 1231838 | **183** | 574.91 | 26302.2 | 3208869 |
| **134** | 420.97 | 14102.6 | 1259833 | **184** | 578.05 | 26590.4 | 3261761 |
| **135** | 424.12 | 14313.9 | 1288249 | **185** | 581.19 | 26880.3 | 3315231 |
| **136** | 427.26 | 14526.7 | 1317090 | **186** | 584.34 | 27171.6 | 3369282 |
| **137** | 430.40 | 14741.1 | 1346357 | **187** | 587.48 | 27464.6 | 3423919 |
| **138** | 433.54 | 14957.1 | 1376055 | **188** | 590.62 | 27759.1 | 3479142 |
| **139** | 436.68 | 15174.7 | 1406187 | **189** | 593.76 | 28055.2 | 3534956 |
| **140** | 439.82 | 15393.8 | 1436755 | **190** | 596.90 | 28352.9 | 3591364 |
| **141** | 442.96 | 15614.5 | 1467763 | **191** | 600.04 | 28652.1 | 3648369 |
| **142** | 446.11 | 15836.8 | 1499214 | **192** | 603.19 | 28952.9 | 3705973 |
| **143** | 449.25 | 16060.6 | 1531112 | **193** | 606.33 | 29255.3 | 3764181 |
| **144** | 452.39 | 16286.0 | 1563457 | **194** | 609.47 | 29559.2 | 3822996 |
| **145** | 455.53 | 16513.0 | 1596256 | **195** | 612.61 | 29864.8 | 3882419 |
| **146** | 458.67 | 16741.5 | 1629511 | **196** | 615.75 | 30171.9 | 3942456 |
| **147** | 461.81 | 16971.7 | 1663224 | **197** | 618.89 | 30480.5 | 4003108 |
| **148** | 464.96 | 17203.4 | 1697398 | **198** | 622.04 | 30790.7 | 4064379 |
| **149** | 468.10 | 17436.6 | 1732038 | **199** | 625.18 | 31102.6 | 4126272 |
| **150** | 471.24 | 17671.5 | 1767146 | **200** | 628.32 | 31415.9 | 4188790 |

## TABLE XII. — SEGMENTS OF THE CIRCLE OF WHICH THE RADIUS IS 1.

| Central Angle | Length of Arc | Rise of Arc | Length of Chord | Area of Segment | Central Angle | Length of Arc | Rise of Arc | Length of Chord | Area of Segment |
|---|---|---|---|---|---|---|---|---|---|
| 1° | 0.0175 | 0.0000 | 0.0175 | 0.00000 | 46° | 0.8029 | 0.0795 | 0.7815 | 0.04176 |
| 2 | 0.0349 | 0.0002 | 0.0349 | 0.00000 | 47 | 0.8203 | 0.0829 | 0.7975 | 0.04448 |
| 3 | 0.0524 | 0.0003 | 0.0524 | 0.00001 | 48 | 0.8378 | 0.0865 | 0.8135 | 0.04731 |
| 4 | 0.0698 | 0.0006 | 0.0698 | 0.00003 | 49 | 0.8552 | 0.0900 | 0.8294 | 0.05025 |
| 5 | 0.0873 | 0.0010 | 0.0872 | 0.00006 | 50 | 0.8727 | 0.0937 | 0.8452 | 0.05331 |
| 6 | 0.1047 | 0.0014 | 0.1047 | 0.00010 | 51 | 0.8901 | 0.0974 | 0.8610 | 0.05649 |
| 7 | 0.1222 | 0.0019 | 0.1221 | 0.00015 | 52 | 0.9076 | 0.1012 | 0.8767 | 0.05978 |
| 8 | 0.1396 | 0.0024 | 0.1395 | 0.00023 | 53 | 0.9250 | 0.1051 | 0.8924 | 0.06319 |
| 9 | 0.1571 | 0.0031 | 0.1569 | 0.00032 | 54 | 0.9425 | 0.1090 | 0.9080 | 0.06673 |
| 10 | 0.1745 | 0.0038 | 0.1743 | 0.00044 | 55 | 0.9599 | 0.1130 | 0.9235 | 0.07039 |
| 11 | 0.1920 | 0.0046 | 0.1917 | 0.00059 | 56 | 0.9774 | 0.1171 | 0.9389 | 0.07417 |
| 12 | 0.2094 | 0.0055 | 0.2091 | 0.00076 | 57 | 0.9948 | 0.1212 | 0.9543 | 0.07808 |
| 13 | 0.2269 | 0.0064 | 0.2264 | 0.00097 | 58 | 1.0123 | 0.1254 | 0.9696 | 0.08212 |
| 14 | 0.2443 | 0.0075 | 0.2437 | 0.00121 | 59 | 1.0297 | 0.1296 | 0.9848 | 0.08629 |
| 15 | 0.2618 | 0.0086 | 0.2611 | 0.00149 | 60 | 1.0472 | 0.1340 | 1.0000 | 0.09059 |
| 16 | 0.2793 | 0.0097 | 0.2783 | 0.00181 | 61 | 1.0647 | 0.1384 | 1.0151 | 0.09502 |
| 17 | 0.2967 | 0.0110 | 0.2956 | 0.00217 | 62 | 1.0821 | 0.1428 | 1.0301 | 0.09958 |
| 18 | 0.3142 | 0.0123 | 0.3129 | 0.00257 | 63 | 1.0996 | 0.1474 | 1.0450 | 0.10428 |
| 19 | 0.3316 | 0.0137 | 0.3301 | 0.00302 | 64 | 1.1170 | 0.1520 | 1.0598 | 0.10911 |
| 20 | 0.3491 | 0.0152 | 0.3473 | 0.00352 | 65 | 1.1345 | 0.1566 | 1.0746 | 0.11408 |
| 21 | 0.3665 | 0.0167 | 0.3645 | 0.00408 | 66 | 1.1519 | 0.1613 | 1.0893 | 0.11919 |
| 22 | 0.3840 | 0.0184 | 0.3816 | 0.00468 | 67 | 1.1694 | 0.1661 | 1.1039 | 0.12443 |
| 23 | 0.4014 | 0.0201 | 0.3987 | 0.00535 | 68 | 1.1868 | 0.1710 | 1.1184 | 0.12982 |
| 24 | 0.4189 | 0.0219 | 0.4158 | 0.00607 | 69 | 1.2043 | 0.1759 | 1.1328 | 0.13535 |
| 25 | 0.4363 | 0.0237 | 0.4329 | 0.00686 | 70 | 1.2217 | 0.1808 | 1.1472 | 0.14102 |
| 26 | 0.4538 | 0.0256 | 0.4499 | 0.00771 | 71 | 1.2392 | 0.1859 | 1.1614 | 0.14683 |
| 27 | 0.4712 | 0.0276 | 0.4669 | 0.00862 | 72 | 1.2566 | 0.1910 | 1.1756 | 0.15279 |
| 28 | 0.4887 | 0.0297 | 0.4838 | 0.00961 | 73 | 1.2741 | 0.1961 | 1.1896 | 0.15889 |
| 29 | 0.5061 | 0.0319 | 0.5008 | 0.01067 | 74 | 1.2915 | 0.2014 | 1.2036 | 0.16514 |
| 30 | 0.5236 | 0.0341 | 0.5176 | 0.01180 | 75 | 1.3090 | 0.2066 | 1.2175 | 0.17154 |
| 31 | 0.5411 | 0.0364 | 0.5345 | 0.01301 | 76 | 1.3265 | 0.2120 | 1.2313 | 0.17808 |
| 32 | 0.5585 | 0.0387 | 0.5513 | 0.01429 | 77 | 1.3439 | 0.2174 | 1.2450 | 0.18477 |
| 33 | 0.5760 | 0.0412 | 0.5680 | 0.01566 | 78 | 1.3614 | 0.2229 | 1.2586 | 0.19160 |
| 34 | 0.5934 | 0.0437 | 0.5847 | 0.01711 | 79 | 1.3788 | 0.2284 | 1.2722 | 0.19859 |
| 35 | 0.6109 | 0.0463 | 0.6014 | 0.01864 | 80 | 1.3963 | 0.2340 | 1.2856 | 0.20573 |
| 36 | 0.6283 | 0.0489 | 0.6180 | 0.02027 | 81 | 1.4137 | 0.2396 | 1.2989 | 0.21301 |
| 37 | 0.6458 | 0.0517 | 0.6346 | 0.02198 | 82 | 1.4312 | 0.2453 | 1.3121 | 0.22045 |
| 38 | 0.6632 | 0.0545 | 0.6511 | 0.02378 | 83 | 1.4486 | 0.2510 | 1.3252 | 0.22804 |
| 39 | 0.6807 | 0.0574 | 0.6676 | 0.02568 | 84 | 1.4661 | 0.2569 | 1.3383 | 0.23578 |
| 40 | 0.6981 | 0.0603 | 0.6840 | 0.02767 | 85 | 1.4835 | 0.2627 | 1.3512 | 0.24367 |
| 41 | 0.7156 | 0.0633 | 0.7004 | 0.02976 | 86 | 1.5010 | 0.2686 | 1.3640 | 0.25171 |
| 42 | 0.7330 | 0.0664 | 0.7167 | 0.03195 | 87 | 1.5184 | 0.2746 | 1.3767 | 0.25990 |
| 43 | 0.7505 | 0.0696 | 0.7330 | 0.03425 | 88 | 1.5359 | 0.2807 | 1.3893 | 0.26825 |
| 44 | 0.7679 | 0.0728 | 0.7492 | 0.03664 | 89 | 1.5533 | 0.2867 | 1.4018 | 0.27675 |
| 45° | 0.7854 | 0.0761 | 0.7654 | 0.03915 | 90° | 1.5708 | 0.2929 | 1.4142 | 0.28540 |

If in any circle $r$ is the length of the radius and $\phi$ is the number of degrees in the central angle, then

(1) Length of the arc, $a = \frac{\phi}{180^\circ}\pi r$. (2) Rise of the arc, $h = r\left(1 - \cos\frac{\phi}{2}\right) = 2r\sin^2\frac{\phi}{4}$.

## TABLE XII.—SEGMENTS OF THE CIRCLE OF WHICH THE RADIUS IS 1.

| Central Angle | Length of Arc | Rise of Arc | Length of Chord | Area of Segment | Central Angle | Length of Arc | Rise of Arc | Length of Chord | Area of Segment |
|---|---|---|---|---|---|---|---|---|---|
| 91° | 1.5882 | 0.2991 | 1.4265 | 0.29420 | 136° | 2.3736 | 0.6254 | 1.8544 | 0.83949 |
| 92 | 1.6057 | 0.3053 | 1.4387 | 0.30316 | 137 | 2.3911 | 0.6335 | 1.8608 | 0.85455 |
| 93 | 1.6232 | 0.3116 | 1.4507 | 0.31226 | 138 | 2.4086 | 0.6416 | 1.8672 | 0.86971 |
| 94 | 1.6406 | 0.3180 | 1.4627 | 0.32152 | 139 | 2.4260 | 0.6498 | 1.8733 | 0.88497 |
| 95 | 1.6580 | 0.3244 | 1.4746 | 0.33093 | 140 | 2.4435 | 0.6580 | 1.8794 | 0.90034 |
| 96 | 1.6755 | 0.3309 | 1.4863 | 0.34050 | 141 | 2.4609 | 0.6662 | 1.8853 | 0.91580 |
| 97 | 1.6930 | 0.3374 | 1.4979 | 0.35021 | 142 | 2.4784 | 0.6744 | 1.8910 | 0.93135 |
| 98 | 1.7104 | 0.3439 | 1.5094 | 0.36008 | 143 | 2.4958 | 0.6827 | 1.8966 | 0.94700 |
| 99 | 1.7279 | 0.3506 | 1.5208 | 0.37009 | 144 | 2.5133 | 0.6910 | 1.9021 | 0.96274 |
| 100 | 1.7453 | 0.3572 | 1.5321 | 0.38026 | 145 | 2.5307 | 0.6993 | 1.9074 | 0.97858 |
| 101 | 1.7628 | 0.3639 | 1.5432 | 0.39058 | 146 | 2.5482 | 0.7076 | 1.9126 | 0.99449 |
| 102 | 1.7802 | 0.3707 | 1.5543 | 0.40104 | 147 | 2.5656 | 0.7160 | 1.9176 | 1.01050 |
| 103 | 1.7977 | 0.3775 | 1.5652 | 0.41166 | 148 | 2.5831 | 0.7244 | 1.9225 | 1.02658 |
| 104 | 1.8151 | 0.3843 | 1.5760 | 0.42242 | 149 | 2.6005 | 0.7328 | 1.9273 | 1.04275 |
| 105 | 1.8326 | 0.3912 | 1.5867 | 0.43333 | 150 | 2.6180 | 0.7412 | 1.9319 | 1.05900 |
| 106 | 1.8500 | 0.3982 | 1.5973 | 0.44439 | 151 | 2.6354 | 0.7496 | 1.9363 | 1.07532 |
| 107 | 1.8675 | 0.4052 | 1.6077 | 0.45560 | 152 | 2.6529 | 0.7581 | 1.9406 | 1.09171 |
| 108 | 1.8850 | 0.4122 | 1.6180 | 0.46695 | 153 | 2.6704 | 0.7666 | 1.9447 | 1.10818 |
| 109 | 1.9024 | 0.4193 | 1.6282 | 0.47844 | 154 | 2.6878 | 0.7750 | 1.9487 | 1.12472 |
| 110 | 1.9199 | 0.4264 | 1.6383 | 0.49008 | 155 | 2.7053 | 0.7836 | 1.9526 | 1.14132 |
| 111 | 1.9373 | 0.4336 | 1.6483 | 0.50187 | 156 | 2.7227 | 0.7921 | 1.9563 | 1.15799 |
| 112 | 1.9548 | 0.4408 | 1.6581 | 0.51379 | 157 | 2.7402 | 0.8006 | 1.9598 | 1.17472 |
| 113 | 1.9722 | 0.4481 | 1.6678 | 0.52586 | 158 | 2.7576 | 0.8092 | 1.9633 | 1.19151 |
| 114 | 1.9897 | 0.4554 | 1.6773 | 0.53806 | 159 | 2.7751 | 0.8178 | 1.9665 | 1.20835 |
| 115 | 2.0071 | 0.4627 | 1.6868 | 0.55041 | 160 | 2.7925 | 0.8264 | 1.9696 | 1.22525 |
| 116 | 2.0246 | 0.4701 | 1.6961 | 0.56289 | 161 | 2.8100 | 0.8350 | 1.9726 | 1.24221 |
| 117 | 2.0420 | 0.4775 | 1.7053 | 0.57551 | 162 | 2.8274 | 0.8436 | 1.9754 | 1.25921 |
| 118 | 2.0595 | 0.4850 | 1.7143 | 0.58827 | 163 | 2.8449 | 0.8522 | 1.9780 | 1.27626 |
| 119 | 2.0769 | 0.4925 | 1.7233 | 0.60116 | 164 | 2.8623 | 0.8608 | 1.9805 | 1.29335 |
| 120 | 2.0944 | 0.5000 | 1.7321 | 0.61418 | 165 | 2.8798 | 0.8695 | 1.9829 | 1.31049 |
| 121 | 2.1118 | 0.5076 | 1.7407 | 0.62734 | 166 | 2.8972 | 0.8781 | 1.9851 | 1.32766 |
| 122 | 2.1293 | 0.5152 | 1.7492 | 0.64063 | 167 | 2.9147 | 0.8868 | 1.9871 | 1.34487 |
| 123 | 2.1468 | 0.5228 | 1.7576 | 0.65404 | 168 | 2.9322 | 0.8955 | 1.9890 | 1.36212 |
| 124 | 2.1642 | 0.5305 | 1.7659 | 0.66759 | 169 | 2.9496 | 0.9042 | 1.9908 | 1.37940 |
| 125 | 2.1817 | 0.5383 | 1.7740 | 0.68125 | 170 | 2.9671 | 0.9128 | 1.9924 | 1.39671 |
| 126 | 2.1991 | 0.5460 | 1.7820 | 0.69505 | 171 | 2.9845 | 0.9215 | 1.9938 | 1.41404 |
| 127 | 2.2166 | 0.5538 | 1.7899 | 0.70897 | 172 | 3.0020 | 0.9302 | 1.9951 | 1.43140 |
| 128 | 2.2340 | 0.5616 | 1.7976 | 0.72301 | 173 | 3.0194 | 0.9390 | 1.9963 | 1.44878 |
| 129 | 2.2515 | 0.5695 | 1.8052 | 0.73716 | 174 | 3.0369 | 0.9477 | 1.9973 | 1.46617 |
| 130 | 2.2689 | 0.5774 | 1.8126 | 0.75144 | 175 | 3.0543 | 0.9564 | 1.9981 | 1.48359 |
| 131 | 2.2864 | 0.5853 | 1.8199 | 0.76584 | 176 | 3.0718 | 0.9651 | 1.9988 | 1.50101 |
| 132 | 2.3038 | 0.5933 | 1.8271 | 0.78034 | 177 | 3.0892 | 0.9738 | 1.9993 | 1.51845 |
| 133 | 2.3213 | 0.6013 | 1.8341 | 0.79497 | 178 | 3.1067 | 0.9825 | 1.9997 | 1.53589 |
| 134 | 2.3387 | 0.6093 | 1.8410 | 0.80970 | 179 | 3.1241 | 0.9913 | 1.9999 | 1.55334 |
| 135° | 2.3562 | 0.6173 | 1.8478 | 0.82454 | 180° | 3.1416 | 1.0000 | 2.0000 | 1.57080 |

(3) Chord, $c = 2r \sin \frac{\phi}{2}$. (4) Area of the segment $= \frac{r^2}{2}\left(\frac{\pi\phi}{180°} - \sin\phi\right)$.

(5) Area of the sector $= \frac{\pi\phi}{360°} r^2$.

## TABLE XIIIa.—NATURAL VALUES OF THE HYPERBOLIC FUNCTION

$\sinh u = \frac{1}{2}\left(e^{u} - e^{-u}\right)$ from $u = 0$ to $u = 5.09$

| $u$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.0000 | 0.0100 | 0.0200 | 0.0300 | 0.0400 | 0.0500 | 0.0600 | 0.0701 | 0.0801 | 0.0901 |
| 0.1 | 0.1002 | 0.1102 | 0.1203 | 0.1304 | 0.1405 | 0.1506 | 0.1607 | 0.1708 | 0.1810 | 0.1911 |
| 0.2 | 0.2013 | 0.2115 | 0.2218 | 0.2320 | 0.2423 | 0.2526 | 0.2629 | 0.2733 | 0.2837 | 0.2941 |
| 0.3 | 0.3045 | 0.3150 | 0.3255 | 0.3360 | 0.3466 | 0.3572 | 0.3678 | 0.3785 | 0.3892 | 0.4000 |
| 0.4 | 0.4108 | 0.4216 | 0.4325 | 0.4434 | 0.4543 | 0.4653 | 0.4764 | 0.4875 | 0.4986 | 0.5098 |
| 0.5 | 0.5211 | 0.5324 | 0.5438 | 0.5552 | 0.5666 | 0.5782 | 0.5897 | 0.6014 | 0.6131 | 0.6248 |
| 0.6 | 0.6367 | 0.6485 | 0.6605 | 0.6725 | 0.6846 | 0.6967 | 0.7090 | 0.7213 | 0.7336 | 0.7461 |
| 0.7 | 0.7586 | 0.7712 | 0.7838 | 0.7966 | 0.8094 | 0.8223 | 0.8353 | 0.8484 | 0.8615 | 0.8748 |
| 0.8 | 0.8881 | 0.9015 | 0.9150 | 0.9286 | 0.9423 | 0.9561 | 0.9700 | 0.9840 | 0.9981 | 1.0122 |
| 0.9 | 1.0265 | 1.0409 | 1.0554 | 1.0700 | 1.0847 | 1.0995 | 1.1144 | 1.1294 | 1.1446 | 1.1598 |
| 1.0 | 1.1752 | 1.1907 | 1.2063 | 1.2220 | 1.2379 | 1.2539 | 1.2700 | 1.2862 | 1.3025 | 1.3190 |
| 1.1 | 1.3356 | 1.3524 | 1.3693 | 1.3863 | 1.4035 | 1.4208 | 1.4382 | 1.4558 | 1.4735 | 1.4914 |
| 1.2 | 1.5095 | 1.5276 | 1.5460 | 1.5645 | 1.5831 | 1.6019 | 1.6209 | 1.6400 | 1.6593 | 1.6788 |
| 1.3 | 1.6984 | 1.7182 | 1.7381 | 1.7583 | 1.7786 | 1.7991 | 1.8198 | 1.8406 | 1.8617 | 1.8829 |
| 1.4 | 1.9043 | 1.9259 | 1.9477 | 1.9697 | 1.9919 | 2.0143 | 2.0369 | 2.0597 | 2.0827 | 2.1059 |
| 1.5 | 2.1293 | 2.1529 | 2.1768 | 2.2008 | 2.2251 | 2.2496 | 2.2743 | 2.2993 | 2.3245 | 2.3499 |
| 1.6 | 2.3756 | 2.4015 | 2.4276 | 2.4540 | 2.4806 | 2.5075 | 2.5346 | 2.5620 | 2.5896 | 2.6175 |
| 1.7 | 2.6456 | 2.6740 | 2.7027 | 2.7317 | 2.7609 | 2.7904 | 2.8202 | 2.8503 | 2.8806 | 2.9112 |
| 1.8 | 2.9422 | 2.9734 | 3.0049 | 3.0367 | 3.0689 | 3.1013 | 3.1340 | 3.1671 | 3.2005 | 3.2341 |
| 1.9 | 3.2682 | 3.3025 | 3.3372 | 3.3722 | 3.4075 | 3.4432 | 3.4792 | 3.5156 | 3.5523 | 3.5894 |
| 2.0 | 3.6269 | 3.6647 | 3.7028 | 3.7414 | 3.7803 | 3.8196 | 3.8593 | 3.8993 | 3.9398 | 3.9806 |
| 2.1 | 4.0219 | 4.0635 | 4.1056 | 4.1480 | 4.1909 | 4.2342 | 4.2779 | 4.3221 | 4.3666 | 4.4117 |
| 2.2 | 4.4571 | 4.5030 | 4.5494 | 4.5962 | 4.6434 | 4.6912 | 4.7394 | 4.7880 | 4.8372 | 4.8868 |
| 2.3 | 4.9370 | 4.9876 | 5.0387 | 5.0903 | 5.1425 | 5.1951 | 5.2483 | 5.3020 | 5.3562 | 5.4109 |
| 2.4 | 5.4662 | 5.5221 | 5.5785 | 5.6354 | 5.6929 | 5.7510 | 5.8097 | 5.8689 | 5.9288 | 5.9892 |
| 2.5 | 6.0502 | 6.1118 | 6.1741 | 6.2369 | 6.3004 | 6.3645 | 6.4293 | 6.4946 | 6.5607 | 6.6274 |
| 2.6 | 6.6947 | 6.7628 | 6.8315 | 6.9009 | 6.9709 | 7.0417 | 7.1132 | 7.1854 | 7.2583 | 7.3319 |
| 2.7 | 7.4063 | 7.4814 | 7.5572 | 7.6338 | 7.7112 | 7.7894 | 7.8683 | 7.9480 | 8.0285 | 8.1098 |
| 2.8 | 8.1919 | 8.2749 | 8.3586 | 8.4432 | 8.5287 | 8.6150 | 8.7021 | 8.7902 | 8.8791 | 8.9689 |
| 2.9 | 9.0596 | 9.1512 | 9.2437 | 9.3371 | 9.4315 | 9.5268 | 9.6231 | 9.7203 | 9.8185 | 9.9177 |
| 3.0 | 10.018 | 10.119 | 10.221 | 10.324 | 10.429 | 10.534 | 10.640 | 10.748 | 10.856 | 10.966 |
| 3.1 | 11.076 | 11.188 | 11.301 | 11.415 | 11.530 | 11.647 | 11.764 | 11.883 | 12.003 | 12.124 |
| 3.2 | 12.246 | 12.369 | 12.494 | 12.620 | 12.747 | 12.876 | 13.006 | 13.137 | 13.269 | 13.403 |
| 3.3 | 13.538 | 13.674 | 13.812 | 13.951 | 14.092 | 14.234 | 14.377 | 14.522 | 14.668 | 14.816 |
| 3.4 | 14.965 | 15.116 | 15.268 | 15.422 | 15.577 | 15.734 | 15.893 | 16.053 | 16.214 | 16.378 |
| 3.5 | 16.543 | 16.709 | 16.877 | 17.047 | 17.219 | 17.392 | 17.567 | 17.744 | 17.923 | 18.103 |
| 3.6 | 18.285 | 18.470 | 18.655 | 18.843 | 19.033 | 19.224 | 19.418 | 19.613 | 19.811 | 20.010 |
| 3.7 | 20.211 | 20.415 | 20.620 | 20.828 | 21.037 | 21.249 | 21.463 | 21.679 | 21.897 | 22.117 |
| 3.8 | 22.339 | 22.564 | 22.791 | 23.020 | 23.252 | 23.486 | 23.722 | 23.961 | 24.202 | 24.445 |
| 3.9 | 24.691 | 24.939 | 25.190 | 25.444 | 25.700 | 25.958 | 26.219 | 26.483 | 26.749 | 27.018 |
| 4.0 | 27.290 | 27.564 | 27.842 | 28.122 | 28.404 | 28.690 | 28.979 | 29.270 | 29.564 | 29.862 |
| 4.1 | 30.162 | 30.465 | 30.772 | 31.081 | 31.393 | 31.709 | 32.028 | 32.350 | 32.675 | 33.004 |
| 4.2 | 33.336 | 33.671 | 34.009 | 34.351 | 34.697 | 35.046 | 35.398 | 35.754 | 36.113 | 36.476 |
| 4.3 | 36.843 | 37.214 | 37.588 | 37.966 | 38.347 | 38.733 | 39.122 | 39.515 | 39.913 | 40.314 |
| 4.4 | 40.719 | 41.129 | 41.542 | 41.960 | 42.382 | 42.808 | 43.238 | 43.673 | 44.112 | 44.555 |
| 4.5 | 45.003 | 45.455 | 45.912 | 46.374 | 46.840 | 47.311 | 47.787 | 48.267 | 48.752 | 49.242 |
| 4.6 | 49.737 | 50.237 | 50.742 | 51.252 | 51.767 | 52.288 | 52.813 | 53.344 | 53.880 | 54.422 |
| 4.7 | 54.969 | 55.522 | 56.080 | 56.643 | 57.213 | 57.788 | 58.369 | 58.955 | 59.548 | 60.147 |
| 4.8 | 60.751 | 61.362 | 61.979 | 62.601 | 63.231 | 63.866 | 64.508 | 65.157 | 65.812 | 66.473 |
| 4.9 | 67.141 | 67.816 | 68.498 | 69.186 | 69.882 | 70.584 | 71.293 | 72.010 | 72.734 | 73.465 |
| 5.0 | 74.203 | 74.949 | 75.702 | 76.463 | 77.232 | 78.008 | 78.792 | 79.584 | 80.384 | 81.192 |

## TABLE XIII*b*.—COMMON LOGARITHMS OF THE HYPERBOLIC FUNCTION

### Sinh *u*, from *u* = 0 to *u* = 5.09

| *u* | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | —∞ | 8.0000 | 3011 | 4772 | 6022 | 6992 | 7784 | 8455 | 9036 | 9548 |
| 0.1 | 9.0007 | 0423 | 0802 | 1152 | 1475 | 1777 | 2060 | 2325 | 2576 | 2814 |
| 0.2 | 9.3039 | 3254 | 3459 | 3656 | 3844 | 4025 | 4199 | 4366 | 4528 | 4685 |
| 0.3 | 9.4836 | 4983 | 5125 | 5264 | 5398 | 5529 | 5656 | 5781 | 5902 | 6020 |
| 0.4 | 9.6136 | 6249 | 6359 | 6468 | 6574 | 6678 | 6780 | 6880 | 6978 | 7074 |
| 0.5 | 9.7169 | 7262 | 7354 | 7444 | 7533 | 7620 | 7707 | 7791 | 7875 | 7958 |
| 0.6 | 9.8039 | 8119 | 8199 | 8277 | 8354 | 8431 | 8506 | 8581 | 8655 | 8728 |
| 0.7 | 9.8800 | 8872 | 8942 | 9012 | 9082 | 9150 | 9218 | 9286 | 9353 | 9419 |
| 0.8 | 9.9485 | 9550 | 9614 | 9678 | 9742 | 9805 | 9868 | 9930 | 9992 | *0053 |
| 0.9 | 0.0114 | 0174 | 0234 | 0294 | 0353 | 0412 | 0470 | 0529 | 0586 | 0644 |
| 1.0 | 0.0701 | 0758 | 0815 | 0871 | 0927 | 0982 | 1038 | 1093 | 1148 | 1203 |
| 1.1 | 0.1257 | 1311 | 1365 | 1419 | 1472 | 1525 | 1578 | 1631 | 1684 | 1736 |
| 1.2 | 0.1788 | 1840 | 1892 | 1944 | 1995 | 2046 | 2098 | 2148 | 2199 | 2250 |
| 1.3 | 0.2300 | 2351 | 2401 | 2451 | 2501 | 2551 | 2600 | 2650 | 2699 | 2748 |
| 1.4 | 0.2797 | 2846 | 2895 | 2944 | 2993 | 3041 | 3090 | 3138 | 3186 | 3234 |
| 1.5 | 0.3282 | 3330 | 3378 | 3426 | 3474 | 3521 | 3569 | 3616 | 3663 | 3711 |
| 1.6 | 0.3758 | 3805 | 3852 | 3899 | 3946 | 3992 | 4039 | 4086 | 4132 | 4179 |
| 1.7 | 0.4225 | 4272 | 4318 | 4364 | 4411 | 4457 | 4503 | 4549 | 4595 | 4641 |
| 1.8 | 0.4687 | 4733 | 4778 | 4824 | 4870 | 4915 | 4961 | 5007 | 5052 | 5098 |
| 1.9 | 0.5143 | 5188 | 5234 | 5279 | 5324 | 5370 | 5415 | 5460 | 5505 | 5550 |
| 2.0 | 0.5595 | 5640 | 5685 | 5730 | 5775 | 5820 | 5865 | 5910 | 5955 | 6000 |
| 2.1 | 0.6044 | 6089 | 6134 | 6178 | 6223 | 6268 | 6312 | 6357 | 6401 | 6446 |
| 2.2 | 0.6491 | 6535 | 6580 | 6624 | 6668 | 6713 | 6757 | 6802 | 6846 | 6890 |
| 2.3 | 0.6935 | 6979 | 7023 | 7067 | 7112 | 7156 | 7200 | 7244 | 7289 | 7333 |
| 2.4 | 0.7377 | 7421 | 7465 | 7509 | 7553 | 7597 | 7642 | 7686 | 7730 | 7774 |
| 2.5 | 0.7818 | 7862 | 7906 | 7950 | 7994 | 8038 | 8082 | 8126 | 8169 | 8213 |
| 2.6 | 0.8257 | 8301 | 8345 | 8389 | 8433 | 8477 | 8521 | 8564 | 8608 | 8652 |
| 2.7 | 0.8696 | 8740 | 8784 | 8827 | 8871 | 8915 | 8959 | 9003 | 9046 | 9090 |
| 2.8 | 0.9134 | 9178 | 9221 | 9265 | 9309 | 9353 | 9396 | 9440 | 9484 | 9527 |
| 2.9 | 0.9571 | 9615 | 9658 | 9702 | 9746 | 9789 | 9833 | 9877 | 9920 | 9964 |
| 3.0 | 1.0008 | 0051 | 0095 | 0139 | 0182 | 0226 | 0270 | 0313 | 0357 | 0400 |
| 3.1 | 1.0444 | 0488 | 0531 | 0575 | 0618 | 0662 | 0706 | 0749 | 0793 | 0836 |
| 3.2 | 1.0880 | 0923 | 0967 | 1011 | 1054 | 1098 | 1141 | 1185 | 1228 | 1272 |
| 3.3 | 1.1316 | 1359 | 1403 | 1446 | 1490 | 1533 | 1577 | 1620 | 1664 | 1707 |
| 3.4 | 1.1751 | 1794 | 1838 | 1881 | 1925 | 1968 | 2012 | 2056 | 2099 | 2143 |
| 3.5 | 1.2186 | 2230 | 2273 | 2317 | 2360 | 2404 | 2447 | 2491 | 2534 | 2578 |
| 3.6 | 1.2621 | 2665 | 2708 | 2752 | 2795 | 2839 | 2882 | 2925 | 2969 | 3012 |
| 3.7 | 1.3056 | 3099 | 3143 | 3186 | 3230 | 3273 | 3317 | 3360 | 3404 | 3447 |
| 3.8 | 1.3491 | 3534 | 3578 | 3621 | 3665 | 3708 | 3752 | 3795 | 3838 | 3882 |
| 3.9 | 1.3925 | 3969 | 4012 | 4056 | 4099 | 4143 | 4186 | 4230 | 4273 | 4317 |
| 4.0 | 1.4360 | 4403 | 4447 | 4490 | 4534 | 4577 | 4621 | 4664 | 4708 | 4751 |
| 4.1 | 1.4795 | 4838 | 4881 | 4925 | 4968 | 5012 | 5055 | 5099 | 5142 | 5186 |
| 4.2 | 1.5229 | 5273 | 5316 | 5359 | 5403 | 5446 | 5490 | 5533 | 5577 | 5620 |
| 4.3 | 1.5664 | 5707 | 5750 | 5794 | 5837 | 5881 | 5924 | 5968 | 6011 | 6055 |
| 4.4 | 1.6098 | 6141 | 6185 | 6228 | 6272 | 6315 | 6359 | 6402 | 6446 | 6489 |
| 4.5 | 1.6532 | 6576 | 6619 | 6663 | 6706 | 6750 | 6793 | 6836 | 6880 | 6923 |
| 4.6 | 1.6967 | 7010 | 7054 | 7097 | 7141 | 7184 | 7227 | 7271 | 7314 | 7358 |
| 4.7 | 1.7401 | 7445 | 7488 | 7531 | 7575 | 7618 | 7662 | 7705 | 7749 | 7792 |
| 4.8 | 1.7836 | 7879 | 7922 | 7966 | 8009 | 8053 | 8096 | 8140 | 8183 | 8226 |
| 4.9 | 1.8270 | 8313 | 8357 | 8400 | 8444 | 8487 | 8530 | 8574 | 8617 | 8661 |
| 5.0 | 1.8704 | 8748 | 8791 | 8835 | 8878 | 8921 | 8965 | 9008 | 9052 | 9095 |

# TABLE XIVa.—NATURAL VALUES OF THE HYPERBOLIC FUNCTION

## $\operatorname{Cosh} u = \frac{1}{2}\left(e^{u} + e^{-u}\right)$ from $u = 0$ to $u = 5.09$

| $u$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 1.0000 | 1.0001 | 1.0002 | 1.0005 | 1.0008 | 1.0013 | 1.0018 | 1.0025 | 1.0032 | 1.0041 |
| 0.1 | 1.0050 | 1.0061 | 1.0072 | 1.0085 | 1.0098 | 1.0113 | 1.0128 | 1.0145 | 1.0162 | 1.0181 |
| 0.2 | 1.0201 | 1.0221 | 1.0243 | 1.0266 | 1.0289 | 1.0314 | 1.0340 | 1.0367 | 1.0395 | 1.0423 |
| 0.3 | 1.0453 | 1.0484 | 1.0516 | 1.0549 | 1.0584 | 1.0619 | 1.0655 | 1.0692 | 1.0731 | 1.0770 |
| 0.4 | 1.0811 | 1.0852 | 1.0895 | 1.0939 | 1.0984 | 1.1030 | 1.1077 | 1.1125 | 1.1174 | 1.1225 |
| 0.5 | 1.1276 | 1.1329 | 1.1383 | 1.1438 | 1.1494 | 1.1551 | 1.1609 | 1.1669 | 1.1730 | 1.1792 |
| 0.6 | 1.1855 | 1.1919 | 1.1984 | 1.2051 | 1.2119 | 1.2188 | 1.2258 | 1.2330 | 1.2402 | 1.2476 |
| 0.7 | 1.2552 | 1.2628 | 1.2706 | 1.2785 | 1.2865 | 1.2947 | 1.3030 | 1.3114 | 1.3199 | 1.3286 |
| 0.8 | 1.3374 | 1.3464 | 1.3555 | 1.3647 | 1.3740 | 1.3835 | 1.3932 | 1.4029 | 1.4128 | 1.4229 |
| 0.9 | 1.4331 | 1.4434 | 1.4539 | 1.4645 | 1.4753 | 1.4862 | 1.4973 | 1.5085 | 1.5199 | 1.5314 |
| 1.0 | 1.5431 | 1.5549 | 1.5669 | 1.5790 | 1.5913 | 1.6038 | 1.6164 | 1.6292 | 1.6421 | 1.6552 |
| 1.1 | 1.6685 | 1.6820 | 1.6956 | 1.7093 | 1.7233 | 1.7374 | 1.7517 | 1.7662 | 1.7808 | 1.7956 |
| 1.2 | 1.8107 | 1.8258 | 1.8412 | 1.8568 | 1.8725 | 1.8884 | 1.9045 | 1.9208 | 1.9373 | 1.9540 |
| 1.3 | 1.9709 | 1.9880 | 2.0053 | 2.0228 | 2.0404 | 2.0583 | 2.0764 | 2.0947 | 2.1132 | 2.1320 |
| 1.4 | 2.1509 | 2.1700 | 2.1894 | 2.2090 | 2.2288 | 2.2488 | 2.2691 | 2.2896 | 2.3103 | 2.3312 |
| 1.5 | 2.3524 | 2.3738 | 2.3955 | 2.4174 | 2.4395 | 2.4619 | 2.4845 | 2.5073 | 2.5305 | 2.5538 |
| 1.6 | 2.5775 | 2.6013 | 2.6255 | 2.6499 | 2.6746 | 2.6995 | 2.7247 | 2.7502 | 2.7760 | 2.8020 |
| 1.7 | 2.8283 | 2.8549 | 2.8818 | 2.9090 | 2.9364 | 2.9642 | 2.9922 | 3.0206 | 3.0492 | 3.0782 |
| 1.8 | 3.1075 | 3.1371 | 3.1669 | 3.1972 | 3.2277 | 3.2585 | 3.2897 | 3.3212 | 3.3530 | 3.3852 |
| 1.9 | 3.4177 | 3.4506 | 3.4838 | 3.5173 | 3.5512 | 3.5855 | 3.6201 | 3.6551 | 3.6904 | 3.7261 |
| 2.0 | 3.7622 | 3.7987 | 3.8355 | 3.8727 | 3.9103 | 3.9483 | 3.9867 | 4.0255 | 4.0647 | 4.1043 |
| 2.1 | 4.1443 | 4.1847 | 4.2256 | 4.2668 | 4.3085 | 4.3507 | 4.3932 | 4.4362 | 4.4797 | 4.5236 |
| 2.2 | 4.5679 | 4.6127 | 4.6580 | 4.7037 | 4.7499 | 4.7966 | 4.8437 | 4.8914 | 4.9395 | 4.9881 |
| 2.3 | 5.0372 | 5.0868 | 5.1370 | 5.1876 | 5.2388 | 5.2905 | 5.3427 | 5.3954 | 5.4487 | 5.5026 |
| 2.4 | 5.5569 | 5.6119 | 5.6674 | 5.7235 | 5.7801 | 5.8373 | 5.8951 | 5.9535 | 6.0125 | 6.0721 |
| 2.5 | 6.1323 | 6.1931 | 6.2545 | 6.3166 | 6.3793 | 6.4426 | 6.5066 | 6.5712 | 6.6365 | 6.7024 |
| 2.6 | 6.7690 | 6.8363 | 6.9043 | 6.9729 | 7.0423 | 7.1123 | 7.1831 | 7.2546 | 7.3268 | 7.3998 |
| 2.7 | 7.4735 | 7.5479 | 7.6231 | 7.6990 | 7.7758 | 7.8533 | 7.9136 | 8.0106 | 8.0905 | 8.1712 |
| 2.8 | 8.2527 | 8.3351 | 8.4182 | 8.5022 | 8.5871 | 8.6728 | 8.7594 | 8.8469 | 8.9352 | 9.0244 |
| 2.9 | 9.1146 | 9.2056 | 9.2976 | 9.3905 | 9.4844 | 9.5791 | 9.6749 | 9.7716 | 9.8693 | 9.9680 |
| 3.0 | 10.068 | 10.168 | 10.270 | 10.373 | 10.477 | 10.581 | 10.687 | 10.794 | 10.902 | 11.011 |
| 3.1 | 11.122 | 11.233 | 11.345 | 11.459 | 11.574 | 11.689 | 11.807 | 11.925 | 12.044 | 12.165 |
| 3.2 | 12.287 | 12.410 | 12.534 | 12.660 | 12.786 | 12.915 | 13.044 | 13.175 | 13.307 | 13.440 |
| 3.3 | 13.575 | 13.711 | 13.848 | 13.987 | 14.127 | 14.269 | 14.412 | 14.556 | 14.702 | 14.850 |
| 3.4 | 14.999 | 15.149 | 15.301 | 15.455 | 15.610 | 15.766 | 15.924 | 16.084 | 16.245 | 16.408 |
| 3.5 | 16.573 | 16.739 | 16.907 | 17.077 | 17.248 | 17.421 | 17.596 | 17.772 | 17.951 | 18.131 |
| 3.6 | 18.313 | 18.497 | 18.682 | 18.870 | 19.059 | 19.250 | 19.444 | 19.639 | 19.836 | 20.035 |
| 3.7 | 20.236 | 20.439 | 20.644 | 20.852 | 21.061 | 21.272 | 21.486 | 21.702 | 21.919 | 22.139 |
| 3.8 | 22.362 | 22.586 | 22.813 | 23.042 | 23.273 | 23.507 | 23.743 | 23.982 | 24.222 | 24.466 |
| 3.9 | 24.711 | 24.959 | 25.210 | 25.463 | 25.719 | 25.977 | 26.238 | 26.502 | 26.768 | 27.037 |
| 4.0 | 27.308 | 27.582 | 27.860 | 28.139 | 28.422 | 28.707 | 28.996 | 29.287 | 29.581 | 29.878 |
| 4.1 | 30.178 | 30.482 | 30.788 | 31.097 | 31.409 | 31.725 | 32.044 | 32.365 | 32.691 | 33.019 |
| 4.2 | 33.351 | 33.686 | 34.024 | 34.366 | 34.711 | 35.060 | 35.412 | 35.768 | 36.127 | 36.490 |
| 4.3 | 36.857 | 37.227 | 37.601 | 37.979 | 38.360 | 38.746 | 39.135 | 39.528 | 39.925 | 40.326 |
| 4.4 | 40.732 | 41.141 | 41.554 | 41.972 | 42.393 | 42.819 | 43.250 | 43.684 | 44.123 | 44.566 |
| 4.5 | 45.014 | 45.466 | 45.923 | 46.385 | 46.851 | 47.321 | 47.797 | 48.277 | 48.762 | 49.252 |
| 4.6 | 49.747 | 50.247 | 50.752 | 51.262 | 51.777 | 52.297 | 52.823 | 53.354 | 53.890 | 54.431 |
| 4.7 | 54.978 | 55.531 | 56.089 | 56.652 | 57.221 | 57.796 | 58.377 | 58.964 | 59.556 | 60.155 |
| 4.8 | 60.759 | 61.370 | 61.987 | 62.609 | 63.239 | 63.874 | 64.516 | 65.164 | 65.819 | 66.481 |
| 4.9 | 67.149 | 67.823 | 68.505 | 69.193 | 69.889 | 70.591 | 71.300 | 72.017 | 72.741 | 73.472 |
| 5.0 | 74.210 | 74.956 | 75.709 | 76.470 | 77.238 | 78.014 | 78.798 | 79.590 | 80.390 | 81.198 |

## TABLE XIVb. — COMMON LOGARITHMS OF THE HYPERBOLIC FUNCTION

### Cosh $u$, from $u = 0$ to $u = 5.09$

| $u$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.0000 | 0000 | 0001 | 0002 | 0003 | 0005 | 0008 | 0011 | 0014 | 0018 |
| 0.1 | 0.0022 | 0026 | 0031 | 0037 | 0042 | 0049 | 0055 | 0062 | 0070 | 0078 |
| 0.2 | 0.0086 | 0095 | 0104 | 0114 | 0124 | 0134 | 0145 | 0156 | 0168 | 0180 |
| 0.3 | 0.0193 | 0205 | 0219 | 0232 | 0246 | 0261 | 0276 | 0291 | 0306 | 0322 |
| 0.4 | 0.0339 | 0355 | 0372 | 0390 | 0407 | 0426 | 0444 | 0463 | 0482 | 0502 |
| 0.5 | 0.0522 | 0542 | 0562 | 0583 | 0605 | 0626 | 0648 | 0670 | 0693 | 0716 |
| 0.6 | 0.0739 | 0762 | 0786 | 0810 | 0835 | 0859 | 0884 | 0910 | 0935 | 0961 |
| 0.7 | 0.0987 | 1013 | 1040 | 1067 | 1094 | 1122 | 1149 | 1177 | 1206 | 1234 |
| 0.8 | 0.1263 | 1292 | 1321 | 1350 | 1380 | 1410 | 1440 | 1470 | 1501 | 1532 |
| 0.9 | 0.1563 | 1594 | 1625 | 1657 | 1689 | 1721 | 1753 | 1785 | 1818 | 1851 |
| 1.0 | 0.1884 | 1917 | 1950 | 1984 | 2018 | 2051 | 2086 | 2120 | 2154 | 2189 |
| 1.1 | 0.2223 | 2258 | 2293 | 2328 | 2364 | 2399 | 2435 | 2470 | 2506 | 2542 |
| 1.2 | 0.2578 | 2615 | 2651 | 2688 | 2724 | 2761 | 2798 | 2835 | 2872 | 2909 |
| 1.3 | 0.2947 | 2984 | 3022 | 3059 | 3097 | 3135 | 3173 | 3211 | 3249 | 3288 |
| 1.4 | 0.3326 | 3365 | 3403 | 3442 | 3481 | 3520 | 3559 | 3598 | 3637 | 3676 |
| 1.5 | 0.3715 | 3754 | 3794 | 3833 | 3873 | 3913 | 3952 | 3992 | 4032 | 4072 |
| 1.6 | 0.4112 | 4152 | 4192 | 4232 | 4273 | 4313 | 4353 | 4394 | 4434 | 4475 |
| 1.7 | 0.4515 | 4556 | 4597 | 4637 | 4678 | 4719 | 4760 | 4801 | 4842 | 4883 |
| 1.8 | 0.4924 | 4965 | 5006 | 5048 | 5089 | 5130 | 5172 | 5213 | 5254 | 5296 |
| 1.9 | 0.5337 | 5379 | 5421 | 5462 | 5504 | 5545 | 5587 | 5629 | 5671 | 5713 |
| 2.0 | 0.5754 | 5796 | 5838 | 5880 | 5922 | 5964 | 6006 | 6048 | 6090 | 6132 |
| 2.1 | 0.6175 | 6217 | 6259 | 6301 | 6343 | 6386 | 6428 | 6470 | 6512 | 6555 |
| 2.2 | 0.6597 | 6640 | 6682 | 6724 | 6767 | 6809 | 6852 | 6894 | 6937 | 6979 |
| 2.3 | 0.7022 | 7064 | 7107 | 7150 | 7192 | 7235 | 7278 | 7320 | 7363 | 7406 |
| 2.4 | 0.7448 | 7491 | 7534 | 7577 | 7619 | 7662 | 7705 | 7748 | 7791 | 7833 |
| 2.5 | 0.7876 | 7919 | 7962 | 8005 | 8048 | 8091 | 8134 | 8176 | 8219 | 8262 |
| 2.6 | 0.8305 | 8348 | 8391 | 8434 | 8477 | 8520 | 8563 | 8606 | 8649 | 8692 |
| 2.7 | 0.8735 | 8778 | 8821 | 8864 | 8907 | 8951 | 8994 | 9037 | 9080 | 9123 |
| 2.8 | 0.9166 | 9209 | 9252 | 9295 | 9338 | 9382 | 9425 | 9468 | 9511 | 9554 |
| 2.9 | 0.9597 | 9641 | 9684 | 9727 | 9770 | 9813 | 9856 | 9900 | 9943 | 9986 |
| 3.0 | 1.0029 | 0073 | 0116 | 0159 | 0202 | 0245 | 0289 | 0332 | 0375 | 0418 |
| 3.1 | 1.0462 | 0505 | 0548 | 0591 | 0635 | 0678 | 0721 | 0764 | 0808 | 0851 |
| 3.2 | 1.0894 | 0938 | 0981 | 1024 | 1067 | 1111 | 1154 | 1197 | 1241 | 1284 |
| 3.3 | 1.1327 | 1371 | 1414 | 1457 | 1501 | 1544 | 1587 | 1631 | 1674 | 1717 |
| 3.4 | 1.1761 | 1804 | 1847 | 1891 | 1934 | 1977 | 2021 | 2064 | 2107 | 2151 |
| 3.5 | 1.2194 | 2237 | 2281 | 2324 | 2367 | 2411 | 2454 | 2497 | 2541 | 2584 |
| 3.6 | 1.2628 | 2671 | 2714 | 2758 | 2801 | 2844 | 2888 | 2931 | 2974 | 3018 |
| 3.7 | 1.3061 | 3105 | 3148 | 3191 | 3235 | 3278 | 3322 | 3365 | 3408 | 3452 |
| 3.8 | 1.3495 | 3538 | 3582 | 3625 | 3669 | 3712 | 3755 | 3799 | 3842 | 3886 |
| 3.9 | 1.3929 | 3972 | 4016 | 4059 | 4103 | 4146 | 4189 | 4233 | 4276 | 4320 |
| 4.0 | 1.4363 | 4406 | 4450 | 4493 | 4537 | 4580 | 4623 | 4667 | 4710 | 4754 |
| 4.1 | 1.4797 | 4840 | 4884 | 4927 | 4971 | 5014 | 5057 | 5101 | 5144 | 5188 |
| 4.2 | 1.5231 | 5274 | 5318 | 5361 | 5405 | 5448 | 5492 | 5535 | 5578 | 5622 |
| 4.3 | 1.5665 | 5709 | 5752 | 5795 | 5839 | 5882 | 5926 | 5969 | 6012 | 6056 |
| 4.4 | 1.6099 | 6143 | 6186 | 6230 | 6273 | 6316 | 6360 | 6403 | 6447 | 6490 |
| 4.5 | 1.6533 | 6577 | 6620 | 6664 | 6707 | 6751 | 6794 | 6837 | 6881 | 6924 |
| 4.6 | 1.6968 | 7011 | 7055 | 7098 | 7141 | 7185 | 7228 | 7272 | 7315 | 7358 |
| 4.7 | 1.7402 | 7445 | 7489 | 7532 | 7576 | 7619 | 7662 | 7706 | 7749 | 7793 |
| 4.8 | 1.7836 | 7880 | 7923 | 7966 | 8010 | 8053 | 8097 | 8140 | 8184 | 8227 |
| 4.9 | 1.8270 | 8314 | 8357 | 8401 | 8444 | 8487 | 8531 | 8574 | 8618 | 8661 |
| 5.0 | 1.8705 | 8748 | 8791 | 8835 | 8878 | 8922 | 8965 | 9009 | 9052 | 9095 |

## TABLE XVa. — NATURAL VALUES OF THE HYPERBOLIC FUNCTION Tanh $u$ from $u = 0$ to $u = 2.39$.

| $u$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.0000 | 0100 | 0200 | 0300 | 0400 | 0500 | 0599 | 0699 | 0798 | 0898 |
| 0.1 | 0.0997 | 1096 | 1194 | 1293 | 1391 | 1489 | 1587 | 1684 | 1781 | 1878 |
| 0.2 | 0.1974 | 2070 | 2165 | 2260 | 2355 | 2449 | 2543 | 2636 | 2729 | 2821 |
| 0.3 | 0.2913 | 3004 | 3095 | 3185 | 3275 | 3364 | 3452 | 3540 | 3627 | 3714 |
| 0.4 | 0.3800 | 3885 | 3969 | 4053 | 4137 | 4219 | 4301 | 4382 | 4462 | 4542 |
| 0.5 | 0.4621 | 4700 | 4777 | 4854 | 4930 | 5005 | 5080 | 5154 | 5227 | 5299 |
| 0.6 | 0.5370 | 5441 | 5511 | 5581 | 5649 | 5717 | 5784 | 5850 | 5915 | 5980 |
| 0.7 | 0.6044 | 6107 | 6169 | 6231 | 6291 | 6352 | 6411 | 6469 | 6527 | 6584 |
| 0.8 | 0.6640 | 6696 | 6751 | 6805 | 6858 | 6911 | 6963 | 7014 | 7064 | 7114 |
| 0.9 | 0.7163 | 7211 | 7259 | 7306 | 7352 | 7398 | 7443 | 7487 | 7531 | 7574 |
| 1.0 | 0.7616 | 7658 | 7699 | 7739 | 7779 | 7818 | 7857 | 7895 | 7932 | 7969 |
| 1.1 | 0.8005 | 8041 | 8076 | 8110 | 8144 | 8178 | 8210 | 8243 | 8275 | 8306 |
| 1.2 | 0.8337 | 8367 | 8397 | 8426 | 8455 | 8483 | 8511 | 8538 | 8565 | 8591 |
| 1.3 | 0.8617 | 8643 | 8668 | 8693 | 8717 | 8741 | 8764 | 8787 | 8810 | 8832 |
| 1.4 | 0.8854 | 8875 | 8896 | 8917 | 8937 | 8957 | 8977 | 8996 | 9015 | 9033 |
| 1.5 | 0.9052 | 9069 | 9087 | 9104 | 9121 | 9138 | 9154 | 9170 | 9186 | 9202 |
| 1.6 | 0.9217 | 9232 | 9246 | 9261 | 9275 | 9289 | 9302 | 9316 | 9329 | 9342 |
| 1.7 | 0.9354 | 9367 | 9379 | 9391 | 9402 | 9414 | 9425 | 9436 | 9447 | 9458 |
| 1.8 | 0.9468 | 9478 | 9488 | 9498 | 9508 | 9518 | 9527 | 9536 | 9545 | 9554 |
| 1.9 | 0.9562 | 9571 | 9579 | 9587 | 9595 | 9603 | 9611 | 9619 | 9626 | 9633 |
| 2.0 | 0.9640 | 9647 | 9654 | 9661 | 9668 | 9674 | 9680 | 9687 | 9693 | 9699 |
| 2.1 | 0.9705 | 9710 | 9716 | 9722 | 9727 | 9732 | 9738 | 9743 | 9748 | 9753 |
| 2.2 | 0.9757 | 9762 | 9767 | 9771 | 9776 | 9780 | 9785 | 9789 | 9793 | 9797 |
| 2.3 | 0.9801 | 9805 | 9809 | 9812 | 9816 | 9820 | 9823 | 9827 | 9830 | 9834 |

## TABLE XVb.—COMMON LOGARITHMS OF THE HYPERBOLIC FUNCTION Tanh $u$ from $u = 0$ to $u = 2.39$.

| $u$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | $-\infty$ | 8.0000 | 3010 | 4770 | 6018 | 6986 | 7776 | 8444 | 9022 | 9531 |
| 0.1 | 8.9986 | *0396 | *0771 | *1115 | *1433 | *1729 | *2004 | *2263 | *2506 | *2736 |
| 0.2 | 9.2953 | 3159 | 3355 | 3542 | 3720 | 3890 | 4053 | 4210 | 4360 | 4505 |
| 0.3 | 9.4644 | 4778 | 4907 | 5031 | 5152 | 5268 | 5381 | 5490 | 5596 | 5698 |
| 0.4 | 9.5797 | 5894 | 5987 | 6078 | 6166 | 6252 | 6336 | 6417 | 6496 | 6573 |
| 0.5 | 9.6648 | 6720 | 6792 | 6861 | 6928 | 6994 | 7058 | 7121 | 7182 | 7242 |
| 0.6 | 9.7300 | 7357 | 7413 | 7467 | 7520 | 7571 | 7622 | 7671 | 7720 | 7767 |
| 0.7 | 9.7813 | 7858 | 7902 | 7945 | 7988 | 8029 | 8069 | 8109 | 8147 | 8185 |
| 0.8 | 9.8222 | 8258 | 8293 | 8328 | 8362 | 8395 | 8428 | 8459 | 8491 | 8521 |
| 0.9 | 9.8551 | 8580 | 8609 | 8637 | 8664 | 8691 | 8717 | 8743 | 8768 | 8793 |
| 1.0 | 9.8817 | 8841 | 8864 | 8887 | 8909 | 8931 | 8952 | 8973 | 8994 | 9014 |
| 1.1 | 9.9034 | 9053 | 9072 | 9090 | 9108 | 9126 | 9144 | 9161 | 9177 | 9194 |
| 1.2 | 9.9210 | 9226 | 9241 | 9256 | 9271 | 9285 | 9300 | 9314 | 9327 | 9341 |
| 1.3 | 9.9354 | 9367 | 9379 | 9391 | 9404 | 9415 | 9427 | 9438 | 9450 | 9460 |
| 1.4 | 9.9471 | 9482 | 9492 | 9502 | 9512 | 9522 | 9531 | 9540 | 9550 | 9558 |
| 1.5 | 9.9567 | 9576 | 9584 | 9592 | 9601 | 9608 | 9616 | 9624 | 9631 | 9639 |
| 1.6 | 9.9646 | 9653 | 9660 | 9666 | 9673 | 9679 | 9686 | 9692 | 9698 | 9704 |
| 1.7 | 9.9710 | 9716 | 9721 | 9727 | 9732 | 9738 | 9743 | 9748 | 9753 | 9758 |
| 1.8 | 9.9763 | 9767 | 9772 | 9776 | 9781 | 9785 | 9789 | 9794 | 9798 | 9802 |
| 1.9 | 9.9806 | 9810 | 9813 | 9817 | 9821 | 9824 | 9828 | 9831 | 9834 | 9838 |
| 2.0 | 9.9841 | 9844 | 9847 | 9850 | 9853 | 9856 | 9859 | 9862 | 9864 | 9867 |
| 2.1 | 9.9870 | 9872 | 9875 | 9877 | 9880 | 9882 | 9884 | 9887 | 9889 | 9891 |
| 2.2 | 9.9893 | 9895 | 9898 | 9900 | 9902 | 9904 | 9905 | 9907 | 9909 | 9911 |
| 2.3 | 9.9913 | 9914 | 9916 | 9918 | 9919 | 9921 | 9923 | 9924 | 9926 | 9927 |

## CONSTANTS.

| | | Numerical Value | Common Logarithm |
|---|---|---|---|
| Ratio of circumference to diameter of a circle, Area of circle of which the radius is 1, Surface of sphere of which the diameter is 1, | $\pi =$ | 3.14159 26535 89793 | 0.4971499 |
| | $\pi^2 =$ | 9.86960 44010 89359 | 0.9942997 |
| | $\pi^3 =$ | 31.00627 66802 93493 | 1.4914496 |
| | $\sqrt{\pi} =$ | 1.77245 38509 05516 | 0.2485749 |
| | $\sqrt[3]{\pi} =$ | 1.46459 18814 91298 | 0.1657166 |
| | $\pi^{-1} =$ | 0.31830 98861 83791 | $\bar{1}$.5028501 |
| The Exponential Base = the base of Natural Logarithms. | $e =$ | 2.71828 18284 59045 | 0.4342945 |
| Modulus of Common Logarithms | $= \log_{10} e = M =$ | 0.43429 44819 03252 | |
| | $\log_e 10 =$ | 2.30258 50929 94046 | |
| | $\log_{10} 2 =$ | 0.30102 99956 63981 | |
| | $\log_e 2 =$ | 0.69314 71805 59945 | |
| | $\log_{10} 3 =$ | 0.47712 12547 19662 | |
| | $\log_e 3 =$ | 1.09861 22886 68110 | |
| | $\log_e \pi =$ | 1.14472 98858 49400 | |
| Radius expressed in degrees of arc | = | 57°.29577 95130 | 1.7581226 |
| in minutes | = | 3437′.74677 07849 | 3.5362739 |
| in seconds | = | 206264″.80624 70964 | 5.3144251 |

| | | |
|---|---|---|
| Arc 1° = $\pi$ : 180 | = 0.01745 32925 19943 | $\bar{2}$.2418774 |
| Arc 1′ = $\pi$ : 10800 | = 0.00029 08882 08666 | $\bar{4}$.4637261 |
| Arc 1″ = $\pi$ : 648000 | = 0.00000 48481 36811 09536 | $\bar{6}$.6855749 |
| sin 1″ | = 0.00000 48481 36811 07637 | |
| tan 1″ | = 0.00000 48481 36811 15234 | |

## WEIGHTS AND MEASURES.

### I. Time.

60 seconds (s.) = 1 minute (m.),
60 minutes = 1 hour (h.),
24 hours = 1 mean solar day (d.).

| | Seconds. | Minutes. | Hours. | Day. |
|---|---|---|---|---|
| Second . . . . . . . . | 1 | | | |
| Minute . . . . . . . . | 60 | 1 | | |
| Hour . . . . . . . . . | 3600 | 60 | 1 | |
| Day . . . . . . . . . | 86400 | 1440 | 24 | 1 |

### II. Arc or Angle.

60 seconds ($''$) = 1 minute ($'$),
60 minutes = 1 degree ($^\circ$),
90 degrees = 1 quadrant,
4 quadrants = 1 circumference = $360^\circ = 21600'$
$= 1296000''$.

### Correspondence of Time and Arc.

24 h. = $360^\circ$ $\qquad$ $1^\circ = 4$ m.
6 h. = $90^\circ$ $\qquad$ $1' = 4$ s.
1 h. = $15^\circ$ $\qquad$ $1'' = \frac{4}{60}$ of one second of time.
1 m. = $15'$
1 s. = $15''$.

### III. Long Measure.

12 inches (in.) = 1 foot (ft.),
3 feet = 1 yard (yd.)
$5\frac{1}{2}$ yards or $16\frac{1}{2}$ feet = 1 rod (rd.),
40 rods = 1 furlong (f.)
8 furlongs = 1 mile (m.).

| | Inches. | Feet. | Yards. | Rods. | Furlongs. | Mile. |
|---|---|---|---|---|---|---|
| Inch . . . . | 1 | | | | | |
| Foot . . . . | 12 | 1 | | | | |
| Yard . . . . | 36 | 3 | 1 | | | |
| Rod . . . . | 198 | 16.5 | 5.5 | 1 | | |
| Furlong . . | 7920 | 660 | 220 | 40 | 1 | |
| Mile . . . . | 63360 | 5280 | 1760 | 320 | 8 | 1 |

A Gunter's Chain, used by surveyors, is 66 feet or 4 rods long, and is divided into 100 links. 80 chains = 1 mile. A fathom = 6 feet.

The standard yard of the United States is a copy of the old English standard yard, and is slightly longer than the Imperial standard yard now used in England.

United States standard yard = 1.000024 Imperial Yards.

A geographical, nautical, or sea mile (also called a knot) is the length of one minute of longitude on the Earth's equator. It is equal to 6086.07 feet = 1.152664 statute or land miles.

1 League = 3 nautical miles.

### IV. Square Measure.

144 square inches (sq. in.) = 1 square foot (sq. ft.),
9 square feet = 1 square yard (sq. yd.),
$30\frac{1}{4}$ square yards or $272\frac{1}{4}$ square feet = 1 square rod (sq. rd.),
160 square rods = 1 acre (A.),
640 acres = 1 square mile (sq. m.).

| | Square Inches. | Square Feet. | Square Yards. | Square Rods. | Acres. | Square Mile. |
|---|---|---|---|---|---|---|
| Sq. in . . . | 1 | | | | | |
| Sq. ft. . . . | 144 | 1 | | | | |
| Sq. yd. . . . | 1296 | 9 | 1 | | | |
| Sq. rd. . . . | . . . | $272\frac{1}{4}$ | $30\frac{1}{4}$ | 1 | | |
| Acre . . . . | . . . . | 43560 | 4840 | 160 | 1 | |
| Sq. mile . . | . . . . | . . . . | . . . . | . . . . | 640 | 1 |

1 acre = 10 square chains.
1 square rod = 625 square links.

### V. Cubic Measure.

1728 cubic inches (cu. in.) = 1 cubic foot (cu. ft.),
27 cubic feet = 1 cubic yard (cu. yd.).

| | Cubic Inches. | Cubic Feet. | Cubic Yards. |
|---|---|---|---|
| Cubic inch . . . . . . | 1 | | |
| Cubic foot . . . . . . . | 1728 | 1 | |
| Cubic yard . . . . . . | 46656 | 27 | 1 |

1 Cord = $4' \times 4' \times 8' = 128$ cubic feet.

1 Cord-foot = $4' \times 4' \times 1' = 16$ cubic feet.

1 Perch of masonry = $16\frac{1}{2}' \times 1\frac{1}{2}' \times 1' = 24.75$ cubic feet, but usually assumed = 25 cubic feet.

### VI. Liquid Measure.

4 gills = 1 pint (pt.) = 28.875 cubic inches,
2 pints = 1 quart (qt.) = 57.75 cubic inches,
4 quarts = 1 gallon (gal.) = 231 cubic inches.

The United States Gallon is the old wine gallon of England, and contains 231 cubic inches.

The Imperial Gallon, the present standard in England, contains 277.274 cubic inches. An imperial gallon of distilled water weighed in air at temperature 62° Fah., barometer at 30 inches, weighs 10 pounds avoirdupois = 70,000 grains.

A United States Gallon of distilled water, weighed under the same conditions, weighs 8.33 pounds avoirdupois, or (more exactly) 58317.798 grains.

A hogshead contains 63 United States gallons, or $52\frac{1}{2}$ Imperial gallons.

The ratio of United States liquid measures to the Imperial liquid measures of the same name is very nearly as 5 to 6.

### VII. Dry Measure.

2 pints = 1 quart (qt.) = 67.2 cubic inches,
8 quarts = 1 peck (pk.) = 537.6 cubic inches,
4 pecks = 1 bushel (bu.) = 2150.42 cubic inches.

The United States Bushel is the Winchester Bushel, formerly the standard in England; and it contains 2150.42 cubic inches.

The Imperial Bushel, the present standard in England, contains 8 Imperial gallons = 2218.192 cubic inches.

**Comparison of English and United States Units of Capacity.**

| | | |
|---|---|---|
| 1 Imperial Gallon = | 277.274 cubic inches = | 1.2003 U. S. Gallons. |
| 1 U. S. Gallon = | 231.000 cubic inches = | 0.8331 Imperial Gallon. |
| I Imperial Bushel = | 2218.192 cubic inches = | 1.0315 U. S. Bushels. |
| 1 U. S. Bushel = | 2150.420 cubic inches = | 0.9695 Imperial Bushels. |

## VIII. Avoirdupois Weights.

| | | Grains. |
|---|---|---|
| 16 drachms (dr.) | = 1 ounce (oz.) | = 437.5, |
| 16 ounces | = 1 pound (lb.) | = 7000, |
| 25 pounds | = 1 quarter (qr.), | |
| 4 quarters | = 1 hundred weight (cwt.), | |
| 20 hundred weight | = 1 ton. | |

The Long Ton or Gross Ton contains 2240 pounds, making the hundred-weight = 112 pounds, and the quarter = 28 pounds.

The principal unit is the pound, which contains 7000 grains. One cubic inch of distilled water weighed in air at temperature 62° Fah., barometer at 30 inches, weighs 252.458 grains. Therefore one avoirdupois pound of matter weighs the same as 27.7274 cubic inches of distilled water weighed under the conditions above stated.

| | | |
|---|---|---|
| Weight of 1 cubic foot of distilled water, temperature 62°, barometer 30 in. | = | in grains, 436247.424<br>in oz. avoir., 997.137<br>in lb. avoir., 62.321 |

Roughly, a cubic foot of water weighs 1000 ounces, or 62½ pounds.

## IX. Troy Weights.

| | |
|---|---|
| 24 grains (gr.) | = 1 pennyweight (dwt.), |
| 20 pennyweights | = 1 ounce (oz.), |
| 12 ounces | = 1 pound (lb.). |

The Troy Pound = 5760 grains.

## X. Apothecaries' Weights.

| | |
|---|---|
| 20 grains (gr.) | = 1 scruple (℈), |
| 3 scruples | = 1 drachm (ʒ), |
| 8 drachms | = 1 ounce (℥). |

The Apothecaries' ounce is identical with the Troy ounce, each contains 480 grains.

The actual standard of weight in the United States is a Troy pound of brass deposited in the United States Mint. It was intended to be an exact copy of the English standard, but has been found to be slightly less heavy.

United States Standard Troy Pound = 0.99999986 Imperial Troy Pound.

## METRIC SYSTEM.

### XI. Long Measure.

Principal unit, the **Meter**, *m.*

| | | |
|---|---|---|
| 1 Myriameter | = | $10{,}000^{m}$ |
| 1 Kilometer (km.) | = | $1{,}000^{m}$ |
| 1 Hectometer | = | $100^{m}$ |
| 1 Dekameter | = | $10^{m}$ |
| 1 METER (m.) | = | $1^{m}$ |
| 1 Decimeter (dm.) | = | $0^{m}.1$ |
| 1 Centimeter (cm.) | = | $0^{m}.01$ |
| 1 Millimeter (mm.) | = | $0^{m}.001$ |

### XII. Square Measure.

Principal unit, the **Are**, *a.*

| | | |
|---|---|---|
| 1 Square Kilometer (km.$^2$) | = | $1{,}000{,}000^{m^2}$ |
| 1 Hectare (ha.) = 100 Are | = | $10{,}000^{m^2}$ |
| 1 ARE (a.) | = | $100^{m^2}$ |
| 1 Centare = 1 Square Meter | = | $1^{m^2}$ |
| 1 Square decimeter (dm.$^2$) | = | $0^{m^2}.01$ |
| 1 Square centimeter (cm.$^2$) | = | $0^{m^2}.0001$ |
| 1 Square millimeter (mm.$^2$) | = | $0^{m^2}.000001$ |

### XIII. Cubic Measure.

**Including all measures of volume or capacity.**

Principal unit, the **Liter**, *l.*

| | | |
|---|---|---|
| 1 Kiloliter = 1 Stere (s.) = 1 Cubic Meter | = | $1000^{l}$ |
| 1 Hectoliter (hl.) | = | $100^{l}$ |
| 1 Dekaliter (dal.) | = | $10^{l}$ |
| 1 LITER (l.) = 1 Cubic Decimeter | = | $1^{l}$ |
| 1 Deciliter (dl.) | = | $0^{l}.1$ |
| 1 Centiliter (cl.) | = | $0^{l}.01$ |
| 1 Milliliter = 1 Microliter ($\lambda$) = 1 Cubic Centimeter | = | $0^{l}.001$ |
| 1 Cubic Millimeter (mm.$^3$) | = | $0^{l}.000001$ |

### XIV. Weights.

Principal units, the **Gram**, *g* and the **Kilogram**, *kg.*

| | | | | |
|---|---|---|---|---|
| 1 M[illegible] = 1 T[illegible] = 1 Metric Ton | = | [illegible]kg | | |
| 1 [illegible] | = | [illegible]kg | | |
| 1 M[illegible] | = | [illegible]kg | | |
| 1 K[illegible] = 1 Kilo | = | [illegible]kg | = | [illegible]g |
| 1 [illegible] | | | = | [illegible]g |
| 1 [illegible] | | | = | [illegible]g |
| 1 [illegible] | | | = | [illegible]g |
| 1 [illegible] | | | = | [illegible] |
| 1 [illegible] | | | = | [illegible] |
| 1 M[illegible] = 1 M[illegible] | | | = | [illegible] |

By Act of Congress,

| | |
|---|---|
| 1 Meter | = 39.37 inches. |
| 1 Kilogram | = 2.2046 pounds avoirdupois, |
| 1 Liter | = 1.0567 quarts, liquid measure |
| | = 0.908 quarts, dry measure. |
| 1 Gram | = 15.4322 grains. |
| 1 Liter | = 61.023 cubic inches. |

## The Force of Gravitation.

The force of gravitation is measured by the velocity imparted to a freely falling body in one second of time. This force is constant for any one place, but varies from place to place on the surface of the earth with the latitude and with the elevation above the sea level. Its computed values at sea level range from $g = 32.088$ feet at the Equator to $g = 32.258$ at the Poles, as shown by the table below, which has been computed by means of Helmert's gravity Formula,

$$g = G\,(1 + 0.005302 \sin^2 l - 0.000007 \sin^2 2l),$$

wherein $G$ is the value of $g$ at the Equator $= 9.78046$ meters $= 32.08807$ feet,

and $l$ is the latitude.

The computed values of gravity at sea level may be corrected for altitude, approximately, by the formula

$$g = g_0\left(1 - 1.32\frac{h}{r}\right),$$

wherein $g_0$ is the value of $g$ at sea level, $h$ is the altitude of the place, and $r$ the mean radius of the Earth = 20,886,852 feet.

### Computed Values of Gravity at Sea Level.

| Latitude. | Values of $g$ in Feet per Second | Latitude. | Value of $g$ in Feet per Second. | Latitude. | Value of $g$ in Feet per Second. |
|---|---|---|---|---|---|
| 0° | 32.088 | | | | |
| 5° | 32.089 | 35° | 32.144 | 65° | 32.228 |
| 10° | 32.093 | 40° | 32.158 | 70° | 32.238 |
| 15° | 32.099 | 45° | 32.173 | 75° | 32.246 |
| 20° | 32.108 | 50° | 32.188 | 80° | 32.253 |
| 25° | 32.118 | 55° | 32.202 | 85° | 32.257 |
| 30° | 32.130 | 60° | 32.215 | 90° | 32.258 |

### Observed Values of Gravity.

Observed values of $g$ for some important places in the United St taken from the Report of the United States Coast and Geodetic Su for 1894 and changed into feet, are given in the following table.

| Place. | Latitude. | | | Height Above Sea Level. | Observed Value of $g$ in Feet. |
|---|---|---|---|---|---|
| Boston. . . . . . . . . . | 42° | 21′ | 33″ | 72 ft. | 32.16470 |
| Philadelphia . . . . . . | 39 | 57 | 06 | 52 | 32.15814 |
| Washington . . . . . . | 38 | 53 | 13 | 46 | 32.15539 |
| Cleveland . . . . . . . | 41 | 30 | 22 | 689 | 32.15962 |
| Cincinnati . . . . . . . | 39 | 08 | 20 | 804 | 32.15184 |
| Chicago . . . . . . . . | 41 | 47 | 25 | 597 | 32.16084 |
| St. Louis. . . . . . . . | 38 | 38 | 03 | 505 | 32.15174 |

At Greenwich, England, latitude 51° 28′ 38″, $g = 32.1908$
At Paris, France, latitude 48 50 11, $g = 32.1843$
At Berlin, Germany, latitude 52 30 17, $g = 32.1937$

The value of $g$ which has been taken as the basis of the Table of Velocities, printed on the next page is $g = 32.16$. If it be desirable to use any other value of $g$ as the basis, the alterations in the values of $v$ can easily be made by means of the numbers given in the column $\Delta_g v$. These numbers are the differences in the value of $v$ due to a change of 0.01 foot in the value of $g$. They are decimals of the same order as the fractional part of the corresponding values of $v$.

### The Seconds Pendulum.

The relation of gravity to the length of the seconds pendulum is expressed by the equation

$$p\pi^2 = g,$$

wherein $p =$ length of pendulum beating seconds,
$\pi^2 = 9.8696044$, $\log_{10} \pi^2 = 0.9942997$.

## VELOCITY, $v$, IN FEET PER SECOND, ACQUIRED BY A BODY FALLING THROUGH $h$ FEET.

$v = \sqrt{2gh}$, wherein $g = 32.16$.

| $h$ | $v$ | $\Delta_g v$ | $h$ | $v$ | $\Delta_g v$ | $h$ | $v$ | $\Delta_g v$ | $h$ | $v$ | $\Delta_g v$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 8.0200 | 12 | 51 | 57.2741 | 89 | 105 | 82.180 | 13 | 425 | 165.336 | 26 |
| 2 | 11.3420 | 18 | 52 | 57.8329 | 90 | 110 | 84.114 | 13 | 450 | 170.129 | 26 |
| 3 | 13.8910 | 22 | 53 | 58.3863 | 91 | 115 | 86.005 | 13 | 475 | 174.791 | 27 |
| 4 | 16.0399 | 25 | 54 | 58.9345 | 92 | 120 | 87.854 | 14 | 500 | 179.332 | 28 |
| 5 | 17.9332 | 28 | 55 | 59.4777 | 93 | 125 | 89.666 | 14 | 525 | 183.761 | 29 |
| 6 | 19.6448 | 31 | 56 | 60.0160 | 93 | 130 | 91.442 | 14 | 550 | 188.085 | 29 |
| 7 | 21.2189 | 33 | 57 | 60.5495 | 94 | 135 | 93.184 | 14 | 575 | 192.312 | 30 |
| 8 | 22.6839 | 35 | 58 | 61.0783 | 95 | 140 | 94.894 | 15 | 600 | 196.448 | 31 |
| 9 | 24.0599 | 37 | 59 | 61.6026 | 96 | 145 | 96.573 | 15 | 625 | 200.499 | 31 |
| 10 | 25.3614 | 39 | 60 | 62.1225 | 97 | 150 | 98.224 | 15 | 650 | 204.470 | 32 |
| 11 | 26.5992 | 41 | 61 | 62.6380 | 97 | 155 | 99.848 | 16 | 675 | 208.365 | 32 |
| 12 | 27.7820 | 43 | 62 | 63.1493 | 98 | 160 | 101.446 | 16 | 700 | 212.189 | 33 |
| 13 | 28.9164 | 45 | 63 | 63.6566 | 99 | 165 | 103.018 | 16 | 725 | 215.944 | 34 |
| 14 | 30.0080 | 47 | 64 | 64.1598 | 100 | 170 | 104.568 | 16 | 750 | 219.636 | 34 |
| 15 | 31.0612 | 48 | 65 | 64.6591 | 100 | 175 | 106.094 | 16 | 775 | 223.267 | 35 |
| 16 | 32.0799 | 50 | 66 | 65.1546 | 101 | 180 | 107.599 | 17 | 800 | 226.839 | 35 |
| 17 | 33.0672 | 51 | 67 | 65.6463 | 102 | 185 | 109.083 | 17 | 825 | 230.356 | 36 |
| 18 | 34.0259 | 53 | 68 | 66.1344 | 103 | 190 | 110.548 | 17 | 850 | 233.820 | 36 |
| 19 | 34.9583 | 54 | 69 | 66.6189 | 104 | 195 | 111.993 | 17 | 875 | 237.234 | 37 |
| 20 | 35.8664 | 56 | 70 | 67.0999 | 104 | 200 | 113.420 | 18 | 900 | 240.599 | 37 |
| 21 | 36.7521 | 57 | 71 | 67.5775 | 105 | 205 | 114.829 | 18 | 925 | 243.918 | 38 |
| 22 | 37.6170 | 58 | 72 | 68.0517 | 106 | 210 | 116.220 | 18 | 950 | 247.192 | 38 |
| 23 | 38.4624 | 60 | 73 | 68.5227 | 107 | 215 | 117.596 | 18 | 975 | 250.424 | 39 |
| 24 | 39.2897 | 61 | 74 | 68.9904 | 108 | 220 | 118.955 | 18 | 1000 | 253.614 | 39 |
| 25 | 40.0999 | 62 | 75 | 69.4550 | 108 | 225 | 120.300 | 19 | 1025 | 256.764 | 40 |
| 26 | 40.8940 | 64 | 76 | 69.9165 | 109 | 230 | 121.629 | 19 | 1050 | 259.877 | 40 |
| 27 | 41.6730 | 65 | 77 | 70.3750 | 109 | 235 | 122.944 | 19 | 1075 | 262.952 | 41 |
| 28 | 42.4377 | 66 | 78 | 70.8305 | 110 | 240 | 124.245 | 19 | 1100 | 265.992 | 41 |
| 29 | 43.1889 | 67 | 79 | 71.2831 | 111 | 245 | 125.532 | 20 | 1125 | 268.998 | 42 |
| 30 | 43.9272 | 68 | 80 | 71.7328 | 112 | 250 | 126.807 | 20 | 1150 | 271.971 | 42 |
| 31 | 44.6533 | 69 | 81 | 72.1798 | 112 | 255 | 128.069 | 20 | 1175 | 274.911 | 43 |
| 32 | 45.3678 | 71 | 82 | 72.6240 | 113 | 260 | 129.318 | 20 | 1200 | 277.820 | 43 |
| 33 | 46.0712 | 72 | 83 | 73.0654 | 114 | 265 | 130.556 | 20 | 1225 | 280.699 | 44 |
| 34 | 46.7641 | 73 | 84 | 73.5043 | 114 | 270 | 131.782 | 20 | 1250 | 283.549 | 44 |
| 35 | 47.4468 | 74 | 85 | 73.9405 | 115 | 275 | 132.996 | 21 | 1275 | 286.370 | 44 |
| 36 | 48.1199 | 75 | 86 | 74.3742 | 116 | 280 | 134.200 | 21 | 1300 | 289.164 | 45 |
| 37 | 48.7836 | 76 | 87 | 74.8054 | 116 | 285 | 135.393 | 21 | 1325 | 291.932 | 45 |
| 38 | 49.4384 | 77 | 88 | 75.2340 | 117 | 290 | 136.575 | 21 | 1350 | 294.673 | 46 |
| 39 | 50.0847 | 78 | 89 | 75.6603 | 118 | 295 | 137.748 | 21 | 1375 | 297.389 | 46 |
| 40 | 50.7228 | 79 | 90 | 76.0842 | 118 | 300 | 138.919 | 22 | 1400 | 300.080 | 47 |
| 41 | 51.3529 | 80 | 91 | 76.5057 | 119 | 310 | 141.206 | 22 | 1500 | 310.612 | 48 |
| 42 | 51.9754 | 81 | 92 | 76.9249 | 120 | 320 | 143.466 | 22 | 1600 | 320.799 | 50 |
| 43 | 52.5905 | 82 | 93 | 77.3418 | 120 | 330 | 145.690 | 23 | 1700 | 330.672 | 51 |
| 44 | 53.1985 | 83 | 94 | 77.7565 | 121 | 340 | 147.881 | 23 | 1800 | 340.259 | 53 |
| 45 | 53.7996 | 84 | 95 | 78.1690 | 122 | 350 | 150.040 | 23 | 1900 | 349.583 | 54 |
| 46 | 54.3941 | 85 | 96 | 78.5794 | 122 | 360 | 152.168 | 24 | 2000 | 358.664 | 56 |
| 47 | 54.9822 | 85 | 97 | 78.9875 | 123 | 370 | 154.267 | 24 | 3000 | 439.272 | 68 |
| 48 | 55.5640 | 86 | 98 | 79.3937 | 123 | 380 | 156.338 | 24 | 4000 | 507.228 | 79 |
| 49 | 56.1398 | 87 | 99 | 79.7977 | 124 | 390 | 158.382 | 25 | 5000 | 567.098 | 88 |
| 50 | 56.7098 | 88 | 100 | 80.1997 | 125 | 400 | 160.399 | 25 | 6000 | 621.225 | 97 |